

On gravitational description of Wilson lines

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ABSTRACT: We study solutions of Type IIB supergravity, which describe the geometries dual to supersymmetric Wilson lines in $\mathcal{N} = 4$ super-Yang-Mills. We show that the solutions are uniquely specified by one function which satisfies a Laplace equation in two dimensions. We show that if this function obeys a certain Dirichlet boundary condition, the corresponding geometry is regular, and we find a simple interpretation of this boundary condition in terms of D3 and D5 branes which are dissolved in the geometry. While all our metrics have $AdS_5 \times S^5$ asymptotics, they generically have nontrivial topologies, which can be uniquely specified by a set of non-contractible three- and five-spheres.

KEYWORDS: AdS-CFT Correspondence, D-branes.

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1. Introduction

According to AdS/CFT correspondence [1, 2], there exists a map between operators in $\mathcal{N} = 4$ SYM and states in string theory on $AdS_5 \times S^5$. This map generically leads to a stringy state on the bulk side, however there is a nice class of BPS operators whose duals are well-described by the type IIB supergravity. Such states have been extensively analyzed in perturbation theory, where one considers linear excitations around $AdS_5 \times S^5$. Computing correlation functions for these perturbations, one finds a remarkable agreement with field theory results (see [3] for the review). However as the conformal weight of operator in field theory becomes large, one should not expect that the linearized solution of supergravity gives a good approximation to the correct geometry, but one can still hope that for a wide class of semiclassical solutions, the stringy corrections are suppressed, and by solving nonlinear equations which follow from the lagrangian of SUGRA, one finds a good description of the bulk state.

A concrete realization of this idea was given in [4], where BPS geometries with $SO(4) \times SO(4)$ symmetry were constructed, and they were shown to have small curvature everywhere. Moreover, the properties of these geometries were in a perfect agreement with expectations coming from field theory, where the BPS states had an effective description in terms of free fermions [5, 6]. The states analyzed in [6] are parameterized in terms of their R charge J and when it is small ($J \ll N$), the dual objects are perturbative gravitons¹ When quantum number J becomes comparable with number of three-branes N , the dual description is given in terms of curved D3 branes which are known as "giant gravitons" [7], and the connection of these branes with field theory was discussed in [8]. Finally, as J becomes much larger than N , the brane probe approximation breaks down, but for certain semiclassical states the geometric description can be trusted (the curvature invariants always remain finite), and corresponding metrics were constructed in [4]. It was shown that various charges computed on the geometric side were in a perfect agreement with corresponding quantities for the Fermi liquid. Moreover, a subsequent work [9, 10] showed that a semiclassical quantization of the geometries led to emergence of free fermions on the gravity side, thus providing a direct map between BPS states in field theory and the moduli space of the geometries.

The goal of this paper is to develop a similar gravitational description for another class of BPS states. In the field theory such states are described by Wilson lines which break one-half of the supersymmetries. It is well-known that in AdS/CFT correspondence, to construct a dual bulk description of a Wilson line one considers an open fundamental string which ends on this line [11, 12]. This picture should be true for the supersymmetric line as well, and this fact seems to imply that fundamental strings would always be present on the bulk side. However it is well-known that in a geometry produced by fundamental string, the dilaton and curvature invariants always diverge at the location of the string, so one concludes that Wilson lines cannot correspond to regular supergravity solutions in the bulk. Thus there seems to be a sharp contrast between these operators and the BPS states studied in [6]: in the latter case the bulk description involved only D3 branes, and the resulting geometries were shown to be smooth [4]. In fact this difference in behavior is only an illusion, and as we review below, the brane configurations dual to Wilson lines should be viewed as D3 branes with fluxes rather than fundamental strings. This picture makes it plausible that in the geometric description, the dilaton stays finite and the metric remains regular, and our construction will show that this is indeed the case. Such "desingularization" is based on the effect discussed in [13], where it was shown that a fundamental string ending on a D3 brane can be viewed as a curved D3 brane which carries electric field (we illustrate it in figure 1). The relevance of this effect for the physics of Wilson lines was first proposed in [11]. The argument of [13] was based on the dynamics of DBI action and to our knowledge its implications for supergravity solutions were never analyzed. In this paper we show that in a certain setup (when supersymmetry

¹In fact, one has to consider an excitation of a coupled system which contains gravitons and five-form flux and but we will call these excitations "gravitons" to be short.

is enhanced by going to the near horizon limit of D3 branes), the effect of [13] leads to regularization of supergravity solution, in particular the dilaton is bounded in the entire space.

In fact the supergravity analysis that we present here was partially performed in a nice paper [14], which was an inspiration for the present work. However [14] gave only several necessary conditions for the geometry to be supersymmetric, and here we solve all supergravity equations. We show that all solutions with $AdS_5 \times S^5$ asymptotics are parameterized by one harmonic function, and if this function obeys certain Neumann boundary conditions, the solution is guaranteed to be regular. Unfortunately to recover the metric, one still needs to solve some differential equations, but we prove that once the harmonic function is specified, this solution exists and it is unique. We also outline a perturbative procedure for constructing the solution.

However, before we discuss supergravity equations it might be useful to recall the description of BPS states in terms of the brane probes. This analysis is presented in section 2. In section 3 we summarize the gravity solution (while the relevant algebra is presented in the appendices), and show that the geometry is regular. Section 4 demonstrates that $AdS_5 \times S^5$ solution can be easily recovered from the general formalism, and in section 5 we construct the perturbative series around this solution. The existence and uniqueness of this series proves that any harmonic function with correct boundary conditions unambiguously leads to the unique regular geometry. In section 6 we point out that once the complete system of equations is derived, it can be used to describe different brane configurations. In particular, in this paper we are interested in solutions with $AdS_2 \times S^2 \times S^4$ factors, but slight modifications of the system make it appropriate for describing geometries with $AdS_4 \times S^2 \times S^2$ factors. Such geometries are produced by backreaction of D5 branes with $AdS_4 \times S^2$ worldvolumes. It is curious that there exists another analytic continuation which maps our system back to itself, but in a different coordinate frame, and we discuss such continuation in section 6 as well. Finally in section 7 we discuss the topology of the solutions and we show that they admit some 3- and 5-cycles, and by wrapping D3 or D5 branes of this cycles we recover the branes discussed in section 2.

2. Wilson lines and brane probes

In this section we will summarize some known facts about the Wilson loops and brane configurations which are dual to them. Our goal is to construct the gravitational dual of supersymmetric time-like Wilson loops in $\mathcal{N} = 4$ SYM. In field theory such operators are specified by a representation R of the gauge group, and are given by the following expressions:

$$W_R(\mathcal{C}) = \text{Tr}_R P \exp \left(i \int_{\mathcal{C}} ds (A_\mu \dot{x}^\mu + \phi_I \dot{y}^I) \right). \tag{2.1}$$

Here the curve \mathcal{C} is a straight line $x^0 = t, y^m = n^m t$ and n^m is a unit vector in R^6 . The choice of this vector breaks $SO(6)$ R-symmetry down to $SO(5)$ which rotates the remaining five scalars. Before we introduced the Wilson loop, $\mathcal{N} = 4$ SYM had $SU(2, 2)$

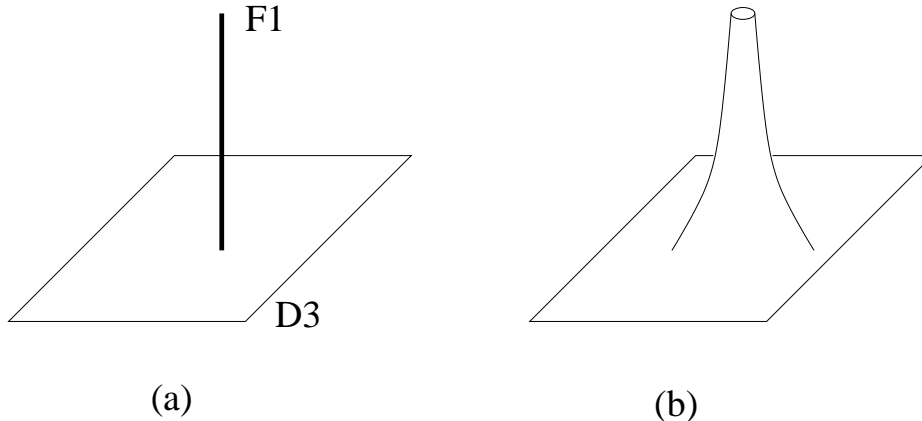


Figure 1: Two different pictures for fundamental string ending on D3 brane: the naive configuration (a) and the description in terms of spike introduced in [13] (b). We will argue that the latter picture is responsible for existence of regular supergravity solution.

conformal symmetry, but the presence of the straight line breaks this symmetry down to $SU(1,1) \times SU(2)$ [15]. Thus we expect that in the presence of Wilson loop, field theory has $SO(6) \times SU(1,1) \times SU(2)$ global symmetry, which implies that the gravity dual would contain AdS_2 , S^2 and S^4 factors².

This is very reminiscent of the situation with BPS chiral primaries discussed in [5, 6]: in that case field theory was defined on $R \times S^3$ and to construct a chiral primary one had to consider zero modes on the sphere. Moreover, a generic 1/2 BPS state broke the R symmetry group down to $SO(4)$ (which is analogous to the $SO(6) \rightarrow SO(5)$ breaking for the Wilson lines), so on the bulk side the relevant symmetry was $SO(4) \times SO(4) \times U(1)$. The geometries for such BPS states were constructed in [4] (and the goal of this paper is to develop a similar picture for the states dual to Wilson lines (2.1)), but before that a great deal of information about the bulk states was extracted in the brane probe approximation. For the BPS states with $SO(4) \times SO(4)$ symmetry the relevant branes were known as "giant gravitons" [7] and they were wrapping cycles either on S^5 (in that case the angular momentum was bounded from above: $J \leq N$) or on AdS_5 . As we will show in this paper, there is a very close analogy between the geometries produced by giant gravitons and the geometries which are dual to the Wilson lines (2.1), so it is very natural to start with discussing the brane configurations which are dual to (2.1).

We begin with the metric written in $AdS_2 \times S^2 \times S^4$ form:

$$ds^2 = R^2 (\cosh^2 \rho dH_2^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2 + d\theta^2 + \sin^2 \theta d\Omega_4^2) \quad (2.2)$$

$$F_5 = 4R^4 (\cosh^2 \rho \sinh^2 \rho d\rho \wedge dH_2 \wedge d\Omega_2 + dual) \quad (2.3)$$

$$R^4 = 4\pi N g. \quad (2.4)$$

According to the proposal of [11, 18], a dual description of the Wilson line is given by a D3 brane with worldvolume $AdS_2 \times S^2$, which is symmetric under $SO(5)$ rotations and has an

²The consequences of $AdS_2 \times S^2$ symmetry for the field theory were recently studied in [16, 14, 17].

electric field along its worldvolume. In an analogy with giant gravitons, we assume that ρ is fixed, this assumption is consistent with equations of motion. Then we look at the action for D3 brane:

$$S_{D3} = -TR^4 \int d^4\sigma \sqrt{(\cosh^4 \rho - E^2) \sinh^4 \rho} + 4TR^4 \int d^4\sigma \int_0^\rho dv \cosh^2 v \sinh^2 v. \quad (2.5)$$

Here E is a value of an electric field on the worldvolume of the brane, to be consistent with symmetries and to have a closed form F_{mn} , this electric field must be constant. To simplify the expressions we defined a rescaled electric field $E = 2\pi R^{-2} F_{01}$. Extremizing the action with respect to ρ , we find an equation

$$-\frac{(1 - 4E^2 + 2 \cosh 2\rho + \cosh 4\rho) \sinh 2\rho}{4\sqrt{-E^2 + \cosh^4 \rho}} + 4 \cosh^2 \rho \sinh^2 \rho = 0. \quad (2.6)$$

This equation is always solved by $\rho = 0$, and now we want to find another solution. Then the relevant equation becomes

$$4(\cosh^2 \rho \cosh 2\rho - E^2) = 4\sqrt{-E^2 + \cosh^4 \rho} \sinh 2\rho$$

and it can easily be solved:

$$E = \cosh \rho. \quad (2.7)$$

We see that the value of the electric field cannot be smaller than one, and if $E = 1$ we are back to the solution $\rho = 0$. For $E > 1$ we find two solutions: (2.7) and $\rho = 0$, they are counterparts of a giant graviton and a usual graviton in [7]. Just as in that case, one can look at a potential for ρ and show that solution $\rho = 0$ is unstable, and the correct expression is (2.7). It may be more convenient to parameterize a brane by an electric displacement Π :

$$\Pi = \frac{\delta S}{\delta E} = \frac{TE}{\sqrt{\cosh^4 \rho - E^2}} \sinh^2 \rho = T \sinh \rho \quad (2.8)$$

which goes to zero as the brane shrinks to zero size. Moreover, the electric displacement controls a coupling of the electric field F_{01} with a bulk Kalb-Ramond field B_{01} [19, 20, 22]: to find this coupling in the linear order, one makes a substitution $2\pi F \rightarrow 2\pi F - B$ in (2.5), then after performing an integration over Ω_2 , one finds a coupling

$$\delta S = \int d^4\sigma \frac{\delta S_{D3}}{\delta(2\pi F_{ab})} \hat{B}_{tx} = \Omega_2 T_3 R^2 \sinh \rho \int dt dx B_{tx} \quad (2.9)$$

Thus we see that, as expected, the three brane with electric flux sources a charge for fundamental string, and to extract the value of this charge, one has to solve equations of motion for B with source (2.9), construct the relevant field strength $H = dB$, and integrate its dual over an appropriate manifold at infinity. Since, by construction, our string is uniformly smeared on S^2 , the manifold relevant for the present case turns out to be S^5 . Notice that going from the NS-NS three form H to its dual and expressing the later in terms of unit sphere S^5 , one introduces an extra factor of R^2 , so we find an

expression for the number of fundamental strings:

$$\Omega_2 T_3 R^4 \sinh \rho = \frac{n_f}{2\pi} : \quad n_f = 4N \sinh \rho \quad (2.10)$$

Since we are working in the units where $\alpha' = 1$, we have the following expressions for the tension and the volume of the sphere:

$$T_3 = \frac{1}{g(2\pi)^3}, \quad \Omega_2 = 4\pi, \quad (2.11)$$

In $AdS_5 \times S^5$ there exists another brane which preserves $AdS_2 \times S^2 \times S^4$ symmetry: it is a D5 brane with worldvolume $AdS_2 \times S^4$. In the Poincare patch, the solution for such probe brane was found in [20] and authors of [17] explored the relation of this brane to Wilson lines (see also [21] for an interesting discussion of non-BPS case). Let us see how such branes would look in global AdS. We again take an worldvolume field strength to be proportional to the volume of AdS space: $F = \frac{ER^2}{2\pi} d^2 H_2$, then the DBI action for D5 branes becomes

$$S_{D5} = -T_5 R^6 \int d^6 \sigma \sqrt{(\cosh^4 \rho - E^2) \sin^8 \theta} + 4T_5 R^6 E \int d^6 \sigma \int^\theta d\phi \sin^4 \phi \quad (2.12)$$

To have six-dimensional branes which preserve S^2 symmetry, one has to set $\rho = 0$, then equation for θ leads to the relation

$$-4 \sin^3 \theta \left(\sqrt{1 - E^2} \cos \theta - E \sin \theta \right) = 0 \quad (2.13)$$

Again, there is an unstable solution $\theta = 0$, and the stable one

$$\theta = \arctan \frac{\sqrt{1 - E^2}}{E},$$

$$\frac{\Pi}{T_5 R^6} = \sin^3 \theta \cos \theta + 4 \int d^6 \sigma \int^\theta d\phi \sin^4 \phi = \frac{3}{2} \left[\theta - E \sqrt{1 - E^2} \right]. \quad (2.14)$$

As before, we can compute the number of fundamental strings generated by this solution. To this end we first find the relevant coupling to the B field:

$$\delta S = \int d^6 \sigma \frac{\delta S_5}{\delta(2\pi F_{ab})} \hat{B}_{tx} = \frac{3}{2} \Omega_4 T_5 R^4 \left[\theta - E \sqrt{1 - E^2} \right] \int dt dx B_{tx}. \quad (2.15)$$

Notice that in this case the strings are smeared over the four-sphere, so one needs to dualize H_3 in six dimensional space (the surface of integration is S^3), so no factors of R are coming from the dualization. The number of strings is

$$n_f = 2\pi \frac{3}{2} \Omega_4 T_5 R^4 \left[\theta - E \sqrt{1 - E^2} \right] = \frac{N}{\pi} \left[\theta - E \sqrt{1 - E^2} \right]. \quad (2.16)$$

Here we used the expressions for the volume of the sphere and for the tension of the brane:

$$\Omega_4 = \frac{8\pi^2}{3}, \quad T_5 = \frac{1}{(2\pi)^5 g}. \quad (2.17)$$

We observe that for a fixed value of displacement Π , solutions of both (2.8) and (2.14) exist, on the other hand, in terms of E there seems to be a nice complementarity: if $E < 1$ we only have the D5 solution, while for $E > 1$ only D3 solution is present. Of course, it is Π , not E that is related to physical observables, so the situation is quite analogous to the giant gravitons: for the same value of Π we have a "giant" (D5 brane) and a "dual giant" (D3 brane). Notice that in the case of spherical branes, the angular momentum of the giant was bounded ($J \leq N$), while for dual giant it was not [7], and we have a similar picture here: the value of n_f in (2.14) is bounded by $N/2$.

The analogy between D5 brane and the giant graviton (and between D3 brane and the dual giant) is also supported by the field theory consideration of [17]. First we recall that in the field theory, the chiral primaries of [6] were constructed by taking various gauge invariant combination of a single $N \times N$ matrix Z . Such combinations can be classified in terms of representations of permutations group S_N [5], in particular the giants correspond to antisymmetric representations and dual giants correspond to symmetric representations of this group [23, 5]. Recently a similar story emerged for the Wilson lines (2.1): they are characterized by representations of the gauge group, and it was shown in [17] that the symmetric representations correspond to D3 branes with fluxes, while antisymmetric representations correspond to D5 branes.

If we put many giant gravitons together, the brane probe approximation would break down and the geometry would no longer be $AdS_5 \times S^5$. A generic configuration of giants would lead to a space with regions of large curvature, so one would need full string theory to describe such a space. However, there exist semiclassical configurations of giants which lead to regular geometries, and their metric would be well approximated by the solutions of type IIB supergravity. For the giant gravitons such solutions were constructed in [4], and now we want to study similar semiclassical geometries for the D3 and D5 branes which were discussed in this section.

3. Summary of supergravity solution

In this section we will outline the procedure for constructing the supergravity solutions, and the details of the computations are provided in the appendices. As we discussed in the previous section, the solution is expected to have AdS_2 , S^2 and S^4 factors, so the metric and five-form are given by

$$\begin{aligned}
 ds^2 &= e^{2A} dH_2^2 + e^{2B} d\Omega_2^2 + e^{2C} d\Omega_4^2 + h_{ij} dx^i dx^j, \quad i, j = \{1, 2\} \\
 F_5 &= df_3 \wedge d\Omega_4 + *_{10}(df_3 \wedge d\Omega_4).
 \end{aligned}
 \tag{3.1}$$

In the brane probe approximation, one set of the Wilson loops was described in terms of D3 branes with spikes of fundamental strings, and such fundamental strings produced NS-NS B field along AdS_2 direction. Since we already have nonzero five-form, the equation

$$d *_{10} (e^\phi F_3) = g F_5 \wedge H_3
 \tag{3.2}$$

implies that there is also a nontrivial RR potential $C_{\mu\nu}^{(2)}$ along the S^2 directions. Three form also sources the dilaton. In principle we could also have the RR form along AdS_2 and

NS-NS form along S^2 , but one can consistently set them to zero. To see this we notice that equations of motion of type IIB supergravity (but not the SUSY variations) are invariant under the change of sign of RR fields. Since we are looking for solutions with S^4 and S^2 factors, we also have a Z_2 symmetry which simultaneously inverts the orientation of these factors.³ Combination of these two symmetries leaves F_5 , $C_S^{(2)}$ and B_{AdS} invariant, but it changes signs of $C^{(0)}$, $C_{AdS}^{(2)}$ and B_S . In the leading order, the fundamental string probe sources only $C_S^{(2)}$ and B_{AdS} , so this discrete symmetry guarantees that we can consistently set $C^{(0)}$, $C_{AdS}^{(2)}$ and B_S to zero. To summarize, in addition to (3.1), we should excite the following fields:

$$H_3 = df_1 \wedge dH_2, \quad F_3 = df_2 \wedge d\Omega_2, \quad e^\phi. \quad (3.3)$$

Notice that we started with string probe and used the symmetry argument to project out the unwanted components of three-forms, but we would have arrived to the same conclusion if we started from the D5-brane picture.

We can now consider the supersymmetry variations of the ansatz (3.1), (3.3), and the details of this analysis are presented in the appendix. Notice that in the initial steps we are essentially repeating the arguments of [14], although we are using more standard notation. In the end we find that geometry can be expressed in terms of three fields G, H, ϕ :

$$ds^2 = e^{2A} dH_2^2 + e^{2B} d\Omega_2^2 + e^{2C} d\Omega_4^2 + \frac{e^{-2\phi}}{e^{2B} + e^{2C}} (dx^2 + dy^2) \quad (3.4)$$

$$F_5 = df_3 \wedge d\Omega_4 + *_{10}(df_3 \wedge d\Omega_4), \quad H_3 = df_1 \wedge dH_2, \quad F_3 = df_2 \wedge d\Omega_2 \quad (3.5)$$

$$e^{2A} = ye^{H-\phi}, \quad e^{2B} = ye^{G-\phi}, \quad e^{2C} = ye^{-G-\phi}, \quad F = \sqrt{e^{2A} - e^{2B} - e^{2C}} \quad (3.6)$$

$$df_1 = -\frac{4e^{2A+\phi/2}}{e^{2A} - e^{2B}} [e^A F d\phi - e^{B+C} * d\phi], \quad (3.7)$$

$$df_2 = \frac{4e^{2B-\phi/2}}{e^{2A} - e^{2B}} [e^B F d\phi - e^{A+C} * d\phi] \quad (3.8)$$

$$e^B e^{-4C} * df_3 = e^A d\left(A - \frac{\phi}{2}\right) + \frac{1}{4} F e^{-\phi/2-2A} df_1 \quad (3.9)$$

These fields satisfy two differential relations

$$d(H - G - 4\phi) = -\frac{2}{y(e^{2B} + e^{2C})} (e^{2C} dy + F e^{B+C-A} dx) \quad (3.10)$$

$$*d \arctan e^G + \frac{1}{2} d \log \frac{e^A - F}{e^A + F} - \frac{1}{2} e^{-\phi/2-2A} df_1 = 0 \quad (3.11)$$

along with integrability conditions coming from (3.7)–(3.9).⁴ Later we will also need an alternative form of equation (3.9), which can be obtained by combining it with other equations in the system:

$$e^A e^{-4C} * df_3 = e^B d\left(B + \frac{\phi}{2}\right) + \frac{1}{4} F e^{\phi/2-2A} df_2. \quad (3.12)$$

³Notice that if invert orientation of only one of the factors, then self-dual F_5 doesn't just flip sign, but it changes in a more complicated way.

⁴Throughout this paper the star is used to denote a Hodge dual in a flat two dimensional space with coordinates (x, y) , and our convention is $*dy = dx$.

Notice that any solution of this system would lead to a supersymmetric geometry, in this sense we found the necessary and sufficient conditions for having a BPS solution. Unfortunately, we were not able to solve this system of equations. However it is clear that the entire problem can be reduced to one differential equation for one function: for example, equation (3.10) allows us to express G and H in terms of derivatives of a function $\psi \equiv H - G - 4\phi$. Then we also know ϕ as a function of ψ , and (3.11) gives a closed differential equation for a single function ψ . Of course, this is a very inefficient way of solving the system, but it demonstrates that we essentially have one scalar degree of freedom.

It is useful to introduce two more functions Ψ_1 and Ψ_2 by performing decomposition

$$-\frac{1}{4}e^{-2A-\phi/2}df_1 = \frac{e^{-\phi}}{e^{2A} - e^{2B}} \left[e^{A+\phi} F d\phi - y * d\phi \right] \equiv \frac{1}{2}(d\Psi_2 + *d\Psi_1). \quad (3.13)$$

Of course there is an ambiguity in defining Ψ_1 and Ψ_2 : one can add an arbitrary harmonic function to Ψ_2 and subtract the dual of this function from Ψ_1 . We will use this ambiguity to impose the boundary condition

$$\Psi_1|_{y=0} = 0, \quad \Psi_1|_{x^2+y^2 \rightarrow \infty} = 0 \quad (3.14)$$

Let us rewrite equation (3.11) in terms of Ψ_1, Ψ_2 :

$$*d \left[\arctan e^G + \Psi_1 \right] + d \left[\frac{1}{2} \log \frac{e^A - F}{e^A + F} + \Psi_2 \right] = 0 \quad (3.15)$$

this implies that there exists a harmonic function Φ such that

$$(\partial_x^2 + \partial_y^2)\Phi = 0 : \quad \arctan e^G + \Psi_1 = \partial_y \Phi, \quad \frac{1}{2} \log \frac{e^A - F}{e^A + F} + \Psi_2 = \partial_x \Phi \quad (3.16)$$

As we argued before, the solution should be completely determined by one function, and we will specify the geometry by choosing Φ ⁵. While we do not have explicit expressions for the metric components in terms of Φ , we still can determine the correct boundary conditions for this function, and in the next section we will outline the perturbative procedure for constructing geometry for any Φ .

Having formulated the local differential equations, we will now discuss the relevant boundary conditions. Since we are looking at solutions which are dual to BPS states in field theory, we expect the geometries to be regular. We recall that another class of BPS geometries was discussed in [4] where it was shown that locally the metric can be expressed in terms of a single harmonic function. Then regularity led to particular

⁵While it is true that geometry is specified in terms of one function, the choice of Φ does not fix a solution uniquely: this function turns out to be invariant under the constant shifts of the dilaton (while keeping G, H, x, y fixed). To deal with this ambiguity we notice that if the dilaton is rescaled as $e^{\phi'} = g e^{\phi}$, then equations (3.7)–(3.11) remain invariant provided that

$$x' = gx, \quad y' = gy, \quad f'_1 = g^{1/2} f_1, \quad f'_2 = g^{-1/2} f_2, \quad f'_3 = f_3$$

In this paper we fix the value of e^{ϕ} at infinity to be one, then Φ indeed corresponds to a unique solution. The string coupling constant can be recovered by making the rescaling written above.

Figure 2: A pictorial representation of the boundary conditions (3.17) on $y = 0$ line: the dark region corresponds to shrinking S^2 and the light regions correspond to contracting S^4 . Since we are looking for solutions with $AdS_5 \times S^5$ asymptotics, the dark segments are contained in a finite region of the line.

boundary conditions for this function. In the present case, we already saw that the solution can also be specified in terms of one harmonic function, and now we will show that regularity imposes very simple boundary conditions for Φ .

The geometry (3.4) has two spheres and the product of their radii is equal to $ye^{-\phi}$, while the ratio of the radii is e^G . So if one of the spheres goes to zero size, then G approaches either positive or negative infinity, while y goes to zero. Since the ambiguity in Ψ_1 was fixed by (3.14), equation (3.16) leads to two kinds of boundary conditions:

$$\begin{aligned} \partial_y \Phi|_{y=0} = \frac{\pi}{2} : \quad & S^4 \text{ shrinks} \\ \partial_y \Phi|_{y=0} = 0 : \quad & S^2 \text{ shrinks} \end{aligned} \tag{3.17}$$

Thus the line $y = 0$ is divided into the set of regions where normal derivative of Φ has a certain value. This is analogous to the picture of [4] where there was a harmonic function with two kinds of Dirichlet boundary conditions in the plane. Pictorially the line $y = 0$ is shown in figure 2, where dark regions correspond to shrinking S^2 and light regions correspond to shrinking S^4 . Let us assume that the dark regions are given by $x_{2m-1} < x < x_{2m}$, then we can find the complete solution of the Laplace equation:

$$\begin{aligned} \Phi &= \frac{\pi y}{2} - \frac{1}{4} \sum \int_{x_{2m-1}}^{x_{2m}} d\xi \log[(x - \xi)^2 + y^2] \\ &= \frac{\pi y}{2} + \frac{1}{4} \sum \left[-2(x - \xi) + 2y \arctan \frac{x - \xi}{y} + (x - \xi) \log[(x - \xi)^2 + y^2] \right]_{x_{2m-1}}^{x_{2m}} \\ \partial_y \Phi &= \frac{\pi}{2} + \frac{1}{2} \sum \left(\arctan \frac{x - x_{2m}}{y} - \arctan \frac{x - x_{2m-1}}{y} \right) \\ \partial_x \Phi &= \frac{1}{4} \sum \log \frac{(x - x_{2m})^2 + y^2}{(x - x_{2m-1})^2 + y^2}. \end{aligned} \tag{3.18}$$

Notice that we can add an arbitrary function of x to Φ , and we fixed this freedom by requiring that the derivative $\partial_x \Phi$ goes to zero as y goes to infinity. Here we also assumed that the dark segments are concentrated in a finite region of $y = 0$ line. This is required for the solution to be asymptotically $AdS_5 \times S^5$, and in this paper we will only be interested in such solutions.

Let us now show that any harmonic function Φ which has boundary conditions (3.17) in the various regions of $y = 0$ line, leads to a regular geometry. It is clear that once we stay away from $y = 0$ and infinity of (x, y) plane, all coefficients in the metric remain finite, so the geometry is regular. So we only need to analyze the behavior of the metric near $y = 0$ and at infinity. As we will see in the next section, the metric based on harmonic function (3.18) approaches $AdS_5 \times S^5$ at infinity, so the space is clearly regular there. We will now analyze the points on the $y = 0$ line.

We begin with vicinity of the points in the dark region (i.e. we look near $(x, y) = (x_0, 0)$, where $x_{2m-1} < x_0 < x_{2m}$). Then we can expand Φ and equations (3.16), (3.10) become

$$\begin{aligned}
 e^G + \Psi_1 &= yq_1(x), & \frac{1}{2} \log \frac{e^A - F}{e^A + F} + \Psi_2 &= q_2(x) \\
 d(H - G - 4\phi) &= -\frac{2dy}{y} - \frac{2Fe^{-A}}{ye^{-G}} dx
 \end{aligned}
 \tag{3.19}$$

The first equation shows that generically $e^G \sim y$, $\Psi_1 \sim y$, then integrability of the last equation implies that in the leading order⁶

$$Fe^{-A} = \tilde{q}_3(x)ye^{-G} = q_3(x) \tag{3.20}$$

Substituting this into the second equation in (3.16), we find that in the leading order, $\Psi_2 = \Psi_2(x)$. Using all this information, we can write the leading contribution to the equation for the dilaton:

$$-\frac{1}{4}e^{-2A-\phi/2}df_1 = q_3(x)d\phi - e^{-H} * d\phi = d\Psi_2 + *d\Psi_1 \equiv dp_1(x) + O(y) \tag{3.21}$$

This leads us to the important conclusion that neither dilaton nor f_1 diverges as we approach $y = 0$. This behavior should be contrasted with gravitational solution for fundamental string which has a divergent dilaton. So the gravity solution confirms the picture which we discussed in the previous section: rather than having the "naked" sources of fundamentals strings, the geometry is described by regular D3 branes and the string charge is mimicked by the fluxes on the brane.

Looking at the equation for f_2 , we find that

$$df_2 \sim -4e^{2B-\phi/2-A+C} * d\phi \sim y * d\phi \tag{3.22}$$

Thus the potential f_2 scales like the volume of S^2 which is necessary for having a regular solution. Finally we analyze the metric. Since dilaton remains finite, we can find the leading expressions for the warp factors:

$$e^{2B} = a_1(x)y, \quad e^{2C} = a_2(x), \quad e^{2A} = a_3(x) \tag{3.23}$$

In other words, the radii of S^4 and AdS_2 remain finite and the metric in (S^2, x, y) sector remains regular:

$$e^{-\phi}ye^G d\Omega_2^2 + \frac{e^{-\phi}}{y(e^G + e^{-G})}(dx^2 + dy^2) \sim e^{-\phi}q(x) [dy^2 + y^2 d\Omega_2^2 + dx^2] \tag{3.24}$$

The vicinity of the points where $\partial_y \Phi = -\frac{\pi}{2}$ can be analyzed in the analogous fashion. The counterparts of the equations (3.19) are

$$\begin{aligned}
 -e^{-G} + \Psi_1 &= yq_1(x), & \frac{1}{2} \log \frac{e^A - F}{e^A + F} + \Psi_2 &= q_2(x) \\
 d(H - G - 4\phi) &= -\frac{2e^{-2G}dy}{y} - \frac{2Fe^{-A}}{ye^G} dx.
 \end{aligned}
 \tag{3.25}$$

⁶To arrive at this conclusion one should also recall that $e^H \geq e^G + e^{-G} \sim y^{-1}$

With trivial modification of the arguments presented above, we find $e^{-G} \sim y$, while the dilaton, fluxes and the *AdS* warp factor remain finite. To show the regularity of the metric we then only need to analyze the (S^4, x, y) sector of the geometry, and these coordinates combine to give a locally flat six dimensional space similar to (3.24). Finally, at the points where both spheres shrink to zero size, the geometry is also regular, and the simplest way to see this is to "zoom in" on such point by rescaling coordinates. Doing this one concludes that (S^2, S^4, x, y) combine to form a patch of flat eight-dimensional space, which proves the regularity of the geometry.

To summarize, we proved that the problem of finding BPS supergravity solution is reduced to solving equations (3.7)–(3.11). We also demonstrated that the solutions can be specified in terms of one function, and the most convenient way to parameterize the solution is to introduce a harmonic function Φ by (3.16). To describe physical situation (e.g. to avoid imaginary values of e^G) this function should satisfy a simple Neumann boundary conditions (3.17) on a line $y = 0$. We showed that any harmonic function which obeys these conditions leads to a regular geometry, in particular, dilaton always remains finite. Unfortunately, in order to translate information from the harmonic function to the geometry one still needs to solve differential equations. In the section 5 we will present an algorithm which allows one to start from any function Φ and construct gravity solution as a perturbative expansion in the value of dilaton. Since dilaton goes to zero at infinity (asymptotically the space is $AdS_5 \times S^5$) and never diverges, we expect that such perturbation theory should give convergent series rather than asymptotic expansion. We present a perturbative procedure for two reasons. First, it is interesting to look at the leading order correction to AdS space. But more importantly, our argument that the solution is completely determined in terms of Φ was somewhat formal, and perturbative expansion proves this statement by construction. But before we construct the perturbative series, it is useful to recover $AdS_5 \times S^5$ space itself.

4. Example: $AdS_5 \times S^5$

Once we found the equations which describe all BPS geometries, it is interesting to see how $AdS_5 \times S^5$ fits into the general story. As we mentioned in the previous section, the solution should be completely specified in terms of one function, and since $AdS_5 \times S^5$ has vanishing dilaton⁷, we expect that this would be the only solution possessing such property. It is instructive to show that this is indeed the case. First we observe that there is an alternative form of equations (3.10), (3.11) (see appendix A for details):

$$\frac{1}{2}d[e^{2A} - e^{2B}] - \frac{1}{2}(e^{2A} + e^{2B})d\phi + \frac{1}{2}Fe^A e^{-\phi/2-2A}df_1 + F^2d\phi = 0 \tag{4.1}$$

$$e^{B+C} * d\left(C - \frac{\phi}{2}\right) - \frac{1}{4}e^{2B}e^{-\phi/2-2A}df_1 - Fe^A d\left(A - \frac{\phi}{2}\right) - \frac{1}{4}F^2e^{-\phi/2-2A}df_1 = 0. \tag{4.2}$$

⁷See footnote 5 for the discussion of constant dilaton

If dilaton is equal to zero, then according to (3.7), f_1 vanishes as well, and equation (4.1) implies that

$$e^{2A} - e^{2B} = L^2, \quad e^{2C} = L^2 - F^2 < L^2 \tag{4.3}$$

with constant L . These equations guarantee that we can parameterize warp factors in terms of two scalar functions, which we call ρ and θ :

$$e^{2A} = L^2 \cosh^2 \rho, \quad e^{2B} = L^2 \sinh^2 \rho, \quad e^{2C} = L^2 \sin^2 \theta \tag{4.4}$$

At this point we know that $y = L^2 \sinh \rho \sin \theta$, however x is still undetermined. To find an expression for it, we need to use the duality relation $dx = *dy$. In particular, we need the expression for $*d\rho$ and $*d\theta$, and those are provided by the equation (4.2):

$$\frac{1}{\sqrt{1 - e^{2C}}} *de^C = e^{-B} de^A : \quad *d\theta = d\rho \tag{4.5}$$

This information allows us to find the expression for x , and plugging it into the general expression for the metric (3.4), we recover $AdS_5 \times S^5$ space⁸:

$$ds^2 = L^2 (\cosh^2 \rho ds_{AdS}^2 + \sinh^2 \rho ds_2^2 + \sin^2 \theta ds_4^2 + d\rho^2 + d\theta^2) \tag{4.6}$$

$$x = L^2 \cosh \rho \cos \theta, \quad y = L^2 \sin \rho \sin \theta. \tag{4.7}$$

Once we know the warp factors as functions of x and y , we can use equations (3.16) to recover the harmonic function Φ which corresponds to $AdS_5 \times S^5$ solution. The result turns out to be in the form (3.18) with only one dark region with $x_2 = -x_1 = L^2$:

$$\Phi = \frac{\pi y}{2} + \frac{1}{4} \left[2y \arctan \frac{x - \xi}{y} + (x - \xi) \log [(x - \xi)^2 + y^2] \right]_{\xi=-L^2}^{\xi=L^2}. \tag{4.8}$$

This is analogous to the way in which $AdS_5 \times S^5$ arose as a “bubbling solution” of [4], where it corresponded to a harmonic function with sources in a circular dark region.

Starting from $AdS_5 \times S^5$ space we can recover the flat space in three different ways, and all of them would correspond to singular limits since one is changing the asymptotics. The first two ways are similar to the recovery of flat space from the bubbling solutions of [4]: we decompactify one of the spheres by taking some point on the $y = 0$ line and rescaling coordinates to zoom in on this point. For example, if we look near the point in the dark strip, then it is metric of S^4 that has to be rescaled by infinite factor (and metric of AdS_2 is rescaled as well), so we end up with space where directions along S^4 and AdS_2 became flat, while sphere S^2 combines with y direction to give an R^3 . Near the point in the light region, S^2 and S^4 exchange roles. The third way to obtain flat space is to look at the vicinity of the point where dark region merges with light one, and in [4] such points led to pp wave metrics. However, in the present case, such nontrivial limit does not exist, and the only way to obtain a regular geometry is to go all the way to flat space by decompactifying AdS_2 and combining (S^2, S^4, x, y) into R^8 .

⁸The change of variables from (ρ, θ) to (x, y) was found before in [14] by starting from $AdS_5 \times S^5$ solution and combining the warp factors to produce y and x coordinates. In contrast to this approach of matching parameters, we derive this solution, and more importantly, we find a connection to the harmonic function.

5. Perturbative solution

While we were not able to solve the differential equations in the complete generality, it might be interesting to look at special cases where they allow some analytic treatment. In the previous section we considered a particular solution corresponding to $AdS_5 \times S^5$ space and it might be interesting to develop a perturbation theory around this solution. While such perturbative solution is interesting by itself (for example, its properties can be compared with CFT computations for Wilson loops), in our case it would play an important role in demonstrating that the gravity solution exists for any function Φ . In section 3 we gave a heuristic argument that all solutions have to be parameterized in terms of a single function and then we claimed that Φ can be viewed as such function. Since we are interested in solutions that asymptote to $AdS_5 \times S^5$, function Φ would be approaching (4.8) as we go to infinity of (x, y) plane, in particular the dilaton would approach zero. Then starting from large values of (x, y) one can start doing perturbation theory in the value of ϕ , and as we show in this section, every harmonic function Φ defines a unique perturbative series. Alternatively, this series can be viewed as an expansion in powers of $1/\sqrt{x^2 + y^2}$ and certainly it has a nonvanishing radius of convergence. While we do not show that the series converges in the entire plane, we expect the metric components and the fluxes to be analytic, so once we show that there is a unique solution in the asymptotic region we expect that it can be unambiguously continued to the upper-half plane, and the resulting solution is guaranteed to be regular by the arguments of section 3.

Let us begin with equations (3.7)–(3.16) and look at them at large values of radial coordinate in (x, y) plane. If space asymptotes to $AdS_5 \times S^5$ (i.e. if all points x_m in (3.18) are bounded as $|x_m| < x_0$), then at large distances function Φ approaches (4.8) and the deviation would lead to a small correction to $AdS_5 \times S^5$. Let us introduce a small parameter ϵ and write all functions as expansion in its powers:

$$G = G_{(0)} + \sum_1^\infty \epsilon^m g_{(m)}, \quad H = H_0 + \sum_1^\infty \epsilon^m h_{(m)}, \quad \phi = \sum_1^\infty \epsilon^m \phi_{(m)}, \quad (5.1)$$

$$\Phi = \Phi_{AdS} + \epsilon(\Phi - \Phi_{AdS}) \equiv \Phi_{(0)} + \epsilon\Phi_{(1)}.$$

Here quantities with subscript zero correspond to $AdS_5 \times S^5$, and by definition, the series for Φ has only one term. Let us look at equation (3.13) in the m -th order:

$$d\Psi_2^{(m)} + *d\Psi_1^{(m)} = xd\phi_{(m)} - y*d\phi_{(m)} + \dots = d(x\phi_{(m)}) - *d(y\phi_{(m)}) + \dots. \quad (5.2)$$

Here dots represent the terms containing expressions with orders between one and $m - 1$. The terms with $\phi_{(m)}$ turned out to be remarkably simple, in particular due to the relation

$$dx = *dy \quad (5.3)$$

we were able to decompose it into exact and co-exact forms. Suppose we solved the equations for all orders up to $m - 1$ -th, then in (5.2) we know all terms represented by dots

explicitly, and we can decompose them into exact and co-exact forms. This implies that⁹

$$\Psi_2^{(m)} = x\phi_{(m)} + \tilde{\Psi}_2^{(m)}, \quad \Psi_1^{(m)} = -y\phi_{(m)} + \tilde{\Psi}_1^{(m)}. \quad (5.4)$$

These expressions should be substituted into the m -th order of equations (3.7)–(3.16), but in addition we should expand the terms with G and H in powers of epsilon. Since terms in orders less than m are known at this point, we can move them to the right hand side, and the contribution of the m -th order is evaluated in the appendix C. In the end we find:

$$\begin{aligned} \frac{g^{(m)}}{s^2 + sh^2} - 2\phi^{(m)} &= \frac{1}{y}\partial_y\Phi^{(m)}, \\ h^{(m)} - g^{(m)} - 4\phi^{(m)} + 8s^2\phi^{(m)} &= -\left\{\frac{2s^2}{y}\partial_y\Phi^{(m)} + \frac{2c^2}{x}\partial_x\Phi^{(m)}\right\}. \end{aligned} \quad (5.5)$$

Here $\Phi^{(m)}$ contains the contributions from lower orders (the only exception is $\Phi^{(1)}$ which was defined in (5.1)), and thus it is known explicitly. We also introduced a shorthand notation:

$$sh = \sinh \rho, \quad ch = \cosh \rho, \quad s = \sin \theta, \quad c = \cos \theta \quad (5.6)$$

and expressions for these quantities can be obtained by inverting (4.7). Equations (5.5) allow us to express m -th order solution in terms of one unknown function (for example, $g^{(m)}$), and to determine this function we need more equations. In particular, we can take the y component of equation (3.10):

$$\partial_y(h^{(m)} - g^{(m)} - 4\phi^{(m)}) = \frac{8y\phi^{(m)}}{s^2 + sh^2} + \frac{4\partial_y\Phi^{(m)}}{s^2 + sh^2} + \Psi_y^{(m)} \quad (5.7)$$

where $\Psi_y^{(m)}$ contains contributions from lower orders. Using the relations

$$\partial_y\theta = \frac{shc}{sh^2 + s^2}, \quad \frac{y}{s^2 + sh^2} = -\partial_y \log c, \quad \frac{1}{s^2}d(s^2 - \log c) = \frac{1}{sc}(2c^2 + 1)d\theta = d \log \frac{s^3}{c}$$

we arrive at the final equation

$$\frac{c}{s}\partial_y \left(\frac{s^3}{c} \frac{g^{(m)}}{2(s^2 + sh^2)} \right) = \frac{1}{4}\partial_y \left\{ \frac{s^2}{y}\partial_y\Phi^{(m)} - \frac{c^2}{x}\partial_x\Phi^{(m)} \right\} - \frac{1}{8}\Psi_y^{(m)}. \quad (5.8)$$

This completes the proof that starting from any harmonic function Φ which has the same asymptotics as Φ_{AdS} , we can construct a unique perturbative series around $AdS_5 \times S^5$ solution, and this series approximates the solution corresponding to Φ at large distances. We also expect that the series for the dilaton converges everywhere and yields the solution corresponding to Φ , while series for G and H should converge away from the line $y = 0$.

⁹Notice that $y\phi_{(m)}$ vanishes at $y = 0$, this means that to be consistent with (3.14), we should work in a gauge where $\tilde{\Psi}^{(m)}$ vanishes there as well. This is especially important for the first order since it allows us to choose $\tilde{\Psi}_1^{(1)} = 0$.

6. Analytic continuations

The goal of this paper is an exploration of supersymmetric geometries with $AdS_2 \times S^2 \times S^4$ symmetries, however once we derived the main result (3.4)–(3.11), it can be used for describing some other solutions as well. In particular, we arrived at $AdS_2 \times S^2 \times S^4$ factors by analyzing the field theory configurations and recalling the observations of [15] that a one-dimensional Wilson line breaks the conformal group in four dimensions down to $SO(2, 2) \times SU(2)$. Similarly, studying domain walls in field theory, one is naturally led to $AdS_3 \times S^1$ split of the four dimensional space. It turns out that $N = 4$ SYM on this space and on $R \times S^3$ has similar structure of BPS states: all of them preserve $SO(4)$ R-symmetry group. In fact, the geometries dual to BPS states on $AdS_3 \times S^1$ were constructed in [4] by making a certain analytic continuation of metrics dual to states in $R \times S^3$. In the present context, it is very natural to ask whether a similar analytic continuation leads to any interesting statements.

By an analogy with analytic continuation of [4], one may think about exchanging AdS_2 and S^2 factors in the solution. This can be accomplished by the following replacements:

$$ds_{AdS}^2 \leftrightarrow -ds_S^2, \quad x \rightarrow ix', \quad y \rightarrow iy', \quad G \rightarrow G' + \frac{\pi i}{2}, \quad H \rightarrow H' + \frac{\pi i}{2}. \quad (6.1)$$

Substituting this into the equations of motion, we find that after rescaling the fluxes f_1 , f_2 and redefining F as

$$f_1 \rightarrow if'_1, \quad f_2 \rightarrow if'_2, \quad F \rightarrow F' \equiv \sqrt{e^{2B'} - e^{2A'} - e^{2C'}} \quad (6.2)$$

we arrive at the system for the real primed variables:

$$d(H' - G' - 4\phi) = -\frac{2}{y'(-e^{2B'} + e^{2C'})}(e^{2C'} dy' + F' e^{B'+C'-A'} dx) \quad (6.3)$$

$$\frac{1}{4}e^{-2A'-\phi/2}df'_1 = \frac{1}{-e^{2A'} + e^{2B'}} \left[e^{A'} F' d\phi - e^{B'+C'} * d\phi \right], \quad (6.4)$$

$$-\frac{1}{4}e^{-2B'+\phi/2}df'_2 = \frac{1}{-e^{2A'} + e^{2B'}} \left[e^{B'} F' d\phi - e^{A'+C'} * d\phi \right] \quad (6.5)$$

$$e^{B'} e^{-4C'} * df'_3 = e^{A'} d \left(A' - \frac{\phi}{2} \right) - \frac{1}{4} F' e^{-\phi/2-2A'} df'_1 \quad (6.6)$$

$$*d \operatorname{arctanh} e^{G'} - \frac{i}{2} d \log \frac{ie^{A'} - F'}{ie^{A'} + F'} + \frac{1}{2} e^{-\phi/2-2A'} df'_1 = 0.$$

Notice that we have not simplified the last equation to make its origin more transparent, but once simplification is done, the factors of i disappear from that equation:

$$\frac{1}{2} * d \log \frac{e^{G'} - 1}{e^{G'} + 1} - d \arctan \frac{e^{A'}}{F'} + \frac{1}{2} e^{-\phi/2-2A'} df'_1 = 0. \quad (6.7)$$

One can worry that the system of equations written above does not make sense in type IIB supergravity since (6.2) seems to suggest that real values of f'_1 lead to imaginary fluxes in the original solution and vice versa. However this is not the case. To see this we recall the expression for the complex three-form G_3 in the original variables

$$G_3 = e^{-\phi/2} H_3 + ie^{\phi/2} F_3 = e^{-\phi/2} df_1 \wedge dH_2 + ie^{\phi/2} df_2 \wedge d\Omega_2. \quad (6.8)$$

In terms of primed variables this expression becomes

$$G_3 = ie^{\phi/2} df'_2 \wedge dH'_2 - e^{-\phi/2} df'_1 \wedge d\Omega'_2. \quad (6.9)$$

Here we used the following conventions for continuing the volume factors ($ds^2_S \leftrightarrow -ds^2_H$ did not specify the continuation uniquely):

$$d\Omega_2 = -idH'_2, \quad dH_2 = id\Omega'_2. \quad (6.10)$$

We see, that in the new variables there is a NS-NS magnetic and RR electric fields, i.e. we ended up with configuration of NS5 and D1 branes which is S-dual to the one we started with. Comparing (6.8) and (6.9), we conclude that the after analytic continuation, the dilaton is $\phi' = -\phi$, which is consistent with S duality.

If double analytic continuation leads us to the equivalent system, one may wonder why the equations (6.3)–(6.7) are different from the original system (3.7)–(3.11). To be precise, there are some similarities: in particular it we write the equations for the fluxes (3.7), (3.8) in terms of $\phi' = -\phi$, then they go into (6.4) and (6.5) after replacements

$$A' \rightarrow B, \quad B' \rightarrow A, \quad \phi' \rightarrow \phi, \quad f'_1 \rightarrow -f_2, \quad f'_2 \rightarrow -f_1. \quad (6.11)$$

Also equation (6.6) goes into (3.12) under the same replacement. However the two remaining equations (6.3), (6.7) look very much different from their counterparts (3.10), (3.11). This difference is an artifact of our coordinate choice: in the original frame we defined $y = e^{B+C+\phi}$, but tracing the fate of y' under the map (6.11), we find¹⁰ that it goes into $e^{A+C+\phi}$, i.e. we have a description of the same system, but in a different coordinate frame. While there is nothing wrong in defining a coordinate $y' = e^{B'+C'+\phi}$, if we do so the points where S^2 shrinks to zero size would be somewhere in the middle of (x', y') plane and now we will discuss the constraints which come from the regularity conditions at these points.

If we start from the system (6.3)–(6.7) and look for regular geometries, we should not allow the AdS space to shrink to zero size, and since the dilaton should also stay finite, we conclude that on the entire line $y = 0$ it is S^2 that shrinks to zero size. As we know from solving the original system, it is impossible to have a nontrivial solution unless radius of S^4 also goes to zero at some points and in the new description this should happen somewhere in the upper half of the plane (i.e. at $y > 0$). Looking at the equation (6.7) we observe the behavior of terms that do not contain f'_1 :

$$S^4 \text{ shrinks : } y = 0, \quad \log \frac{e^{G'} - 1}{e^{G'} + 1} = 0 \quad (6.12)$$

$$S^2 \text{ shrinks : } \arctan \frac{e^{A'}}{F'} = 0.$$

To take into account the flux, we decompose it as in (3.13) and define the harmonic function Φ' as in (3.16):

$$(\partial_{x'}^2 + \partial_{y'}^2)\Phi' = 0 : \quad \frac{1}{2} \log \frac{e^{G'} - 1}{e^{G'} + 1} + \Psi'_1 = \partial_{y'}\Phi', \quad \arctan \frac{e^{A'}}{F'} + \Psi'_2 = \partial_{x'}\Phi'. \quad (6.13)$$

¹⁰Notice that in the Appendix A we used a combination of (A.26) to argue that y was a convenient coordinate. Alternatively, we could add (A.17) to (A.19), this would naturally lead to y' .

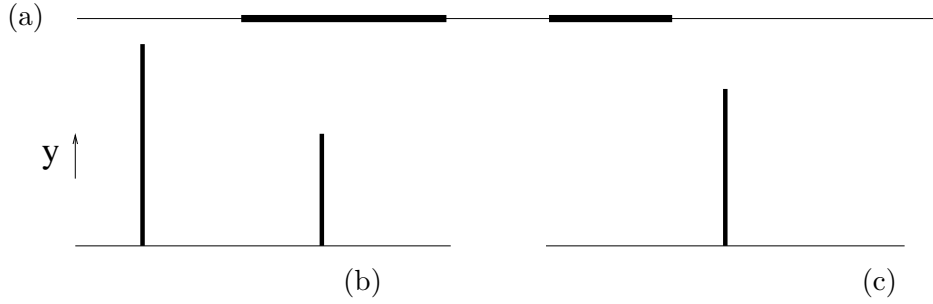


Figure 3: A pictorial representation of the boundary conditions (3.17) on (x, y) plane (a) and the lines on which their counterparts (6.16) are imposed (b). Figure (c) gives an example of such lines which correspond to $AdS_5 \times S^5$.

Since functions Ψ'_1, Ψ'_2 are defined only up to one harmonic function, we can choose this function in a way which makes the boundary conditions (6.12) especially convenient

$$\begin{aligned} S^4 \text{ shrinks : } & y' = 0, \quad \partial_{y'}\Phi' = 0, \quad \Psi'_1 = 0 \\ S^2 \text{ shrinks : } & f(x', y') = 0, \quad \partial_x\Phi' = 0, \quad \Psi'_2 = 0. \end{aligned} \quad (6.14)$$

Now we have to find the restriction on a curve $f(x', y') = 0$. Let us consider some point on this curve where $y' \neq 0$ (otherwise, both S^2 and S^4 shrink to zero and such special points require a separate consideration), then we can rewrite equations (6.3) as

$$\begin{aligned} \partial_{y'}e^{A'} &= e^{A'} \left[-\partial_{y'}(C' - \phi) - \frac{e^{2B'}}{y'(e^{2C'} - e^{2B'})} \right], \\ \partial_{x'}e^{A'} &= -e^{A'}\partial_{y'}(C' - \phi) - \frac{F'}{y'(e^{2C'} - e^{2B'})}. \end{aligned} \quad (6.15)$$

Since we are considering the point where $e^{A'} = 0$, these two equations imply the the gradient of $e^{A'}$ points along x' direction. In other words, the curves of $e^{A'} = 0$ are located at the fixed value of x' . Then we have to impose the boundary conditions along the straight lines depicted in figure 3b:

$$\begin{aligned} S^4 \text{ shrinks : } & y' = 0, \quad \partial_{y'}\Phi' = 0, \quad \Psi'_1 = 0 \\ S^2 \text{ shrinks : } & x' = x_i, \quad 0 < y' < y_i, \quad \partial_{x'}\Phi' = 0, \quad \Psi'_2 = 0. \end{aligned} \quad (6.16)$$

Thus the solution is parameterized by a set of pairs (x_i, y_i) , then one has to solve the Laplace equation with boundary conditions (6.16). The resulting harmonic function Φ' leads to a unique geometry which is guaranteed to be regular.

As in section 4, we can show that setting three-form to zero, we end up with a unique solution, which describes $AdS_5 \times S^5$. Rather than repeating those arguments here, we just state the result that $AdS_5 \times S^5$ corresponds to the boundary conditions along the curve depicted in figure 3c. Notice that the junction of the vertical and horizontal lines

is universal (one can rescale the coordinates to zoom in on this point), and $AdS_5 \times S^5$ example demonstrates that no additional singularity develops at such junction.¹¹

To summarize, we showed that performing a double analytic continuation (6.1) which exchanges AdS_2 and S^2 , one arrives at an alternative description of the same system which uses a different coordinate frame. However in that frame one can also formulate very simple boundary conditions (6.16), so we have two equivalent ways of looking at geometries with $AdS_2 \times S^2 \times S^4$ factors. We now discuss another analytic continuation which leads to a description of new geometries.

In the geometry (3.4) we have two spheres, so one can perform one more continuation:

$$\begin{aligned} ds_{AdS_2}^2 &\rightarrow -ds_{S^2}^2, & x &\rightarrow ix', & G &\rightarrow G' - \frac{\pi i}{2}, & H &\rightarrow H' + \frac{\pi i}{2}, & f_1 &\rightarrow if'_1 \\ ds_{S^4}^2 &\rightarrow -ds_{AdS_4}^2, & y &\rightarrow iy', & & & & & f_3 &\rightarrow if'_3 \end{aligned} \quad (6.17)$$

The resulting geometry is governed by the equations:

$$d(H' - G' - 4\phi) = -\frac{2}{y'(e^{2B'} - e^{2C'})}(-e^{2C'} dy' + F' e^{B'+C'-A'} dx') \quad (6.18)$$

$$\begin{aligned} -\frac{1}{4}e^{-2A'-\phi/2}df'_1 &= \frac{e^{-\phi}}{e^{2A'} + e^{2B'}} \left[e^{A'+\phi} F' d\phi - y' * d\phi \right], \\ \frac{1}{4}e^{-2B'+\phi/2}df'_2 &= \frac{1}{e^{2A'} + e^{2B'}} \left[-e^{B'} F d\phi - e^{A'+C'} * d\phi \right] \\ e^{B'} e^{-4C'} * df'_3 &= e^{A'} d \left(A' - \frac{\phi}{2} \right) - \frac{1}{4} F' e^{-\phi/2-2A'} df'_1 \\ *d \arctan(-ie^{G'}) &+ \frac{1}{2} d \log \frac{ie^{A'} - F'}{ie^{A'} + F'} + \frac{i}{2} e^{-\phi/2-2A'} df'_1 = 0 \end{aligned} \quad (6.19)$$

$$F' \equiv \sqrt{e^{2C'} - e^{2B'} - e^{2A'}} \quad (6.20)$$

For the reference we also give a complete set of SUGRA fields for this case:

$$\begin{aligned} ds^2 &= e^{2A'} (d\Omega'_2)^2 + e^{2B'} d\Omega_2^2 + e^{2C'} (dH'_4)^2 + \frac{e^{-2\phi}}{e^{2C'} - e^{2B'}} ((dx')^2 + (dy')^2) \\ F_5 &= df'_3 \wedge dH'_4 + *_{10}(df'_3 \wedge dH'_4), & H_3 &= df'_1 \wedge d\Omega'_2, & F_3 &= df_2 \wedge d\Omega_2 \\ e^{2A'} &= y' e^{H'-\phi}, & e^{2B'} &= y' e^{G'-\phi}, & e^{2C'} &= y' e^{-G'-\phi}, \end{aligned} \quad (6.21)$$

This time the harmonic function Φ' is defined by

$$(\partial_{x'}^2 + \partial_{y'}^2)\Phi' = 0 : \quad \frac{1}{2} \log \frac{1 - e^{G'}}{e^{G'} + 1} + \Psi'_1 = \partial_{y'}\Phi', \quad -\arctan \frac{e^{A'}}{F'} + \Psi'_2 = \partial_{x'}\Phi' \quad (6.22)$$

Notice that for this continuation we again have to impose the boundary conditions at $y' = 0$ (where one of the S^2 's goes to zero size), and on certain lines $x' = x_0, y' > y_0$ similar to

¹¹It is interesting to observe that the boundary conditions on surfaces similar to ones depicted in figure 3b were encountered in a description of BPS geometries in M theory [24]. However, unlike the present case which has simple Neumann boundary conditions on such curves, the boundary conditions discussed in [24] were more complicated (in that case there was one more coordinate and the "lines" $x = x_i$ actually represented the disks).

what we had in (6.16):

$$\begin{aligned}
 S^2 \text{ shrinks : } & y' = 0, \quad \partial_{y'}\Phi' = 0, \quad \Psi'_1 = 0 \\
 \tilde{S}^2 \text{ shrinks : } & x' = x_i, \quad y' > y_i > 0, \quad \partial_{x'}\Phi' = 0, \quad \Psi'_2 = 0
 \end{aligned}
 \tag{6.23}$$

Of course, there is an alternative way of solving the system with $AdS_4 \times S^2 \times S^2$ factors which is based on introducing a more convenient coordinate $\tilde{y} = e^{A'+B'-\phi'}$, but we will not explore this further.

7. Back to the brane probes

As we showed in section 3, to find geometries with $AdS_2 \times S^2 \times S^4$ factors, one needs to solve a system (3.4)–(3.11). Although we were not able to find new nontrivial solutions of this system, we demonstrated that for the spaces which asymptote to $AdS_5 \times S^5$, the geometries are uniquely parameterized by one harmonic function Φ . Now we want to study some qualitative properties of the solutions and show that they are in a perfect agreement with expectations from the brane probe analysis which was presented in section 2.

We begin with discussing the topology of the solutions. Let us consider a generic boundary condition depicted in figure 2. In the light region, the S^4 shrinks to zero size, so it is useful to take a contour depicted in 4a and construct a five dimensional manifold as a warped product of this contour and S^4 . Restricting metric (3.4) to this manifold, we find that as y approaches zero, the volume of S^4 goes to zero as well and near such points metric is approximated by

$$ds_5^2 = \mathcal{F}(dy^2 + y^2 d\Omega_4^2) \tag{7.1}$$

and it looks like a north pole of S^5 . We conclude that the five manifold that we just described has a topology of S^5 , moreover if there is a dark region between the two endpoints of the contour, (as in figure 4a), this S^5 is not contractible. This is analogous to S^5 which emerged in [4] by combining three dimensional sphere and a certain two-dimensional surface. Moreover, for the $AdS_5 \times S^5$ solution, the five sphere which we described here and the one discussed in [4] are the same, and they simply correspond to the S^5 factor in the geometry.

Since we have a non-contractible five-manifold and there is a nontrivial five-form field strength, it leads to a non-zero flux over such manifold. We see that the dark strips on the $y = 0$ line serve as sources of D3 branes, and by the symmetry arguments one can see that these branes have $AdS_2 \times S^2$ worldvolume. Then we conclude that the dark strips describe a gravitational backreaction of D3 branes which were discussed in section 2.

A similar analysis can be performed for contours which end in the dark regions (see figure 4b). In this case we take a contour and fibrate S^2 over it, then we arrive at a non-contractible three-manifold which has a topology of S^3 . Since we have a magnetic RR three-form, it can have a non-zero flux over such manifold. Then we conclude that the light strips describe gravity solutions for the polarized D5 branes which were discussed in section 2. Of course, the language of D3 branes and D5 branes is only appropriate when we have

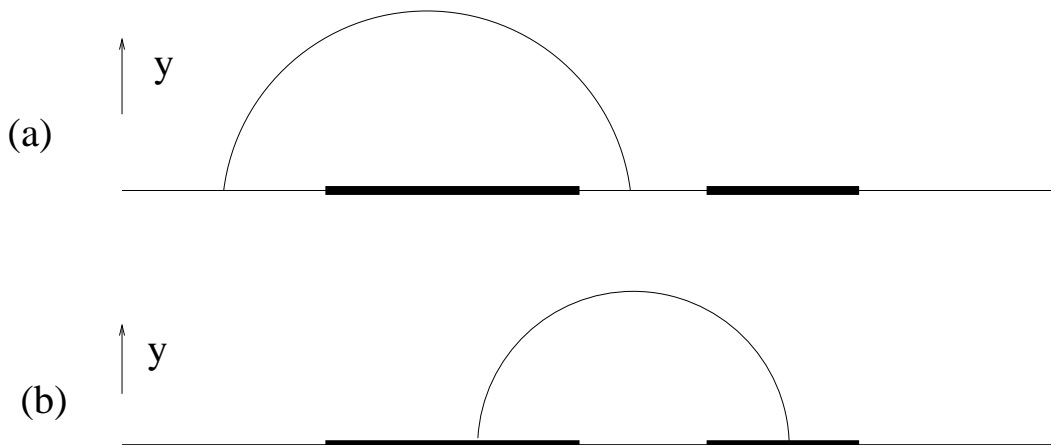


Figure 4: The geometries described in this paper have non-trivial topologies which are characterized by non-contractible 3- and 5-cycles. To construct a five-cycle, one looks at a contour depicted in figure (a) and fibrates S^4 over it. The three-cycles are constructed in a similar ways using contours from figure b and S^2 .

a small dark strip inside a long light region or vice versa, otherwise one has a background where all fluxes are turned on and are comparable in strength. As far as topology of the solution is concerned, we conclude that it is completely determined by the topology of the dark regions on $y = 0$ line, and thus it is uniquely specified by the set of non-contractible three- and five-cycles.

8. Discussion

While we have a very good understanding of supersymmetric branes in flat space, the picture is less clear for the branes in curved spacetimes. Starting from the original discovery of giant gravitons [7], there was a remarkable progress in understanding branes in AdS spaces [25], but most of the work was devoted to studying the brane probe approximation. Such branes are usually curved and to stabilize their shape, they are either moving or have some fluxes on the worldvolume, and the stabilization happens via interaction between such fluxes and background RR field. It would be nice to understand the geometries produced by such curved branes and for the giant gravitons of [7] this problem was solved in [4]. In this paper we looked at another class of 1/2 BPS branes which are supported by fluxes rather than angular momentum and we showed that, as in [4], the geometries are parameterized by one harmonic function with very simple boundary conditions. Unfortunately, to translate this harmonic function into the explicit metric, one still has to solve certain differential equations and we showed that such solution is unique. This is in sharp contrast to a situation in [4] where one starts from a harmonic function and recovers the geometry by simple algebraic manipulations. It would be nice if better understanding of equations (3.7)–(3.16) could lead to a similar picture for our geometries as well.

In this paper we discussed only the branes which preserve half of supersymmetries in $AdS_5 \times S^5$, just as [4] dealt with 1/2 BPS states but with different bosonic symmetries.

It would be interesting to understand the geometries which preserve less supersymmetries, especially since they have a very nice description in the brane probe approximation. For example, giant gravitons that preserve 1/4 and 1/8 of the supersymmetries, are described in terms of the holomorphic surfaces [26]. In particular, such giant gravitons still preserve $S^3 \times R$ symmetry which comes from the AdS part of the geometry, but another $SO(4)$ (which was crucial for the construction of [4]) is broken. Unfortunately the problem of finding the gravity solutions for such branes reduces to a complicated equation of the Monge-Ampere type, and it is not clear what can be learned from it. However, on the field theory side, the interesting progress was made in [27], where it was argued that metrics could arise as semiclassical limit of matrix models. Although so far this approach has not led to any explicit solutions, this direction appears to be very promising.

Recently, the analog of giant gravitons preserving 8 supersymmetries was discussed in the context of branes which are dual Wilson lines [28], and such objects are expected to preserve $AdS_2 \times S^2$ symmetries. It would be interesting to study the gravitational description of such objects in a way similar to the one that we discussed here.

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A. Solving gravity equations

The main goal of this paper is to find supersymmetric geometries which contain $AdS_2 \times S^2 \times S^4$ factors. The motivation for doing this was given in section 2, and in this appendix we give some technical steps which led to the final solution (3.4)–(3.11).

A.1 Formulation of the problem

We are looking for supersymmetric solutions of type IIB supergravity, so we begin with summarizing the fermionic variations using the standard notation of [29]:

$$\begin{aligned} \delta\lambda &= i\not{P}\epsilon^* - \frac{i}{24}\gamma^{mnp}G_{mnp}\epsilon = 0 \\ \delta\psi_M &= \left(\nabla_M - \frac{i}{2}Q_M\right)\epsilon + \frac{i}{480}\not{F}_5\gamma_M\epsilon + \frac{1}{96}(-\gamma_M\not{G} - 2\not{G}\gamma_M)\epsilon^* \end{aligned} \quad (A.1)$$

Supersymmetry parameter ϵ is a complex Weyl spinor ($\Gamma_{11}\epsilon = -\epsilon$), and the expressions for two vectors Q_m, P_m and a scalar B can be found in [29] (see also [30]). Below we will write such expressions for a special case.

Equations (A.1) give SUSY variations for any bosonic background of type IIB SUGRA, but we will need a truncated version of these equations. As argued in section 3, we are interested in solutions with vanishing axion $C^{(0)}$, this implies that $\tau = ie^{-\phi}$, $Q_\mu = 0$, and

$$P_\mu = \left(1 - \left[\frac{1 - e^{-\phi}}{1 + e^{-\phi}}\right]^2\right)^{-1} \partial_\mu \frac{1 - e^{-\phi}}{1 + e^{-\phi}} = \frac{(1 + e^{-\phi})^2}{4e^{-\phi}} \partial_\mu \frac{2}{1 + e^{-\phi}} = \frac{1}{2} \partial_\mu \phi \quad (\text{A.2})$$

$$B = \frac{1 - e^{-\phi}}{1 + e^{-\phi}}, \quad f^{-2} = \frac{4e^{-\phi}}{(1 + e^{-\phi})^2},$$

$$G_3 = f(H_3 + iF_3 - BH_3 + iBF_3) = e^{-\phi/2} H_3 + ie^{\phi/2} F_3 \quad (\text{A.3})$$

Substituting these expressions into (A.1), we arrive at the equations which will be analyzed in the remaining part of this appendix:

$$\delta\lambda = i \not{\partial}\phi\epsilon^* - \frac{i}{24} \gamma^{mnp} G_{mnp} \epsilon = 0 \quad (\text{A.4})$$

$$\delta\psi_M = \nabla_M \epsilon + \frac{i}{480} \not{F}_5 \gamma_M \epsilon + \frac{1}{96} (-\gamma_M \not{G} - 2 \not{G} \gamma_M) \epsilon^* = 0 \quad (\text{A.5})$$

The metric and fluxes are given by equations (3.1), (3.3) and it might be useful to reproduce them here:

$$ds^2 = e^{2A} dH_2^2 + e^{2B} d\Omega_2^2 + e^{2C} d\Omega_4^2 + h_{ij} dx^i dx^j \quad (\text{A.6})$$

$$F_5 = df_3 \wedge d\Omega_4 + *_{10}(df_3 \wedge d\Omega_4), \quad H_3 = df_1 \wedge dH_2, \quad F_3 = df_2 \wedge d\Omega_2, \quad e^\phi \quad (\text{A.7})$$

Equations (A.6) guarantee that all bosonic fields have the required symmetry, but we also need to impose the symmetry on the spinor ϵ . To do this we need to review a construction of spinors on even-dimensional spheres (spinors on AdS are trivial modifications of those) and we devote Appendix B to such review. Here we just summarize the results. Let us look at a covariant derivative ∇_m along one of the directions of S^2 and rewrite it in terms of covariant derivative $\tilde{\nabla}_m$ on a unit two-sphere:

$$\nabla_m \epsilon = \tilde{\nabla}_m \epsilon - \frac{1}{2} \gamma^\mu{}_m \partial_\mu B \quad (\text{A.8})$$

In the appendix B it is shown that the derivative on a unit sphere can be written in terms of hermitean matrix P_S which anticommutes with chirality operator on S^2 and with gamma matrices along the direction orthogonal to this sphere¹²:

$$\tilde{\nabla}_m \epsilon = -\frac{i}{2} e^{-B} \gamma_m P_S \epsilon \quad (\text{A.9})$$

We can now write the complete derivative of the spinor along S^2 direction as well as derivatives along S^4 and AdS_2 :

$$\begin{aligned} S^2 : \quad \nabla_m &= -\frac{1}{2} \gamma_m (ie^{-B} P_S - \not{\partial} B) \\ AdS_2 : \quad \nabla_m &= -\frac{1}{2} \gamma_m (-e^{-A} P_H - \not{\partial} A), \\ S^4 : \quad \nabla_m &= -\frac{1}{2} \gamma_m (ie^{-C} P_\Omega - \not{\partial} C). \end{aligned} \quad (\text{A.10})$$

¹²In that appendix we always considered reduced gamma matrices $\tilde{\gamma}_m$, while here we are writing the ten-dimensional ones. This explains an extra factor of e^{-B} in (A.9) compared to (B.20).

The final ingredient which is needed to write down the equations is the expressions for the fluxes. Looking at the formula for the five form flux and using projection $\Gamma_{11}\epsilon = -\epsilon$, we observe that $\mathcal{F}_5\epsilon$ can be expressed in terms of f_3 and we don't have to evaluate the dual piece:

$$\mathcal{F}_5\epsilon = 2 \times 5! \times e^{-4C} \not{\partial} f_3 \Gamma_\Omega \epsilon, \quad \frac{1}{480} \mathcal{F}_5\epsilon = \frac{e^{-4C}}{2} \not{\partial} f_3 \Gamma_\Omega \epsilon. \quad (\text{A.11})$$

Here Γ_Ω is a hermitean chirality matrix on S^4 , and we also introduce analogous matrices on S^2 (calling it Γ_S) and AdS_2 (it will be denoted Γ_H). Notice that the equations (A.1) are formulated in a basis where all gamma matrices are real, this implies that Γ_S is imaginary, while Γ_H, Γ_Ω are real. While we will not use an explicit form of gamma matrices, their reality and symmetry properties will be important. And rather than summarizing these properties in words, we write an explicit basis of gamma matrices which satisfies all the requirements, so the reader can consult this equation:

$$\begin{aligned} P_H &= \sigma_2 \otimes \sigma_2 \otimes 1_4, & P_S &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes 1_2, & P_\Omega &= \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_2 \\ \Gamma_H &= 1_2 \otimes \sigma_3 \otimes 1_4, & \Gamma_S &= 1_4 \otimes \sigma_2 \otimes 1_2, & \Gamma_\Omega &= 1_8 \otimes \sigma_3, & \Gamma_{1,2} &= \sigma_{3,1} \otimes 1_8 \\ \Gamma_{11} &= -i\Gamma_1\Gamma_2\Gamma_\Omega\Gamma_S\Gamma_H = \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_3 \end{aligned} \quad (\text{A.12})$$

A straightforward computation leads to the expression for the three-form flux:

$$\frac{1}{24} \mathcal{G} = -\frac{1}{4} (e^{\phi/2-2B} \not{\partial} f_2 \Gamma_S + e^{-\phi/2-2A} \not{\partial} f_1 \Gamma_H) \quad (\text{A.13})$$

The first term in the right hand side of this equation is pure imaginary, while the second term is real. Since at some point we will need to do complex conjugation, it is useful to define two matrices¹³:

$$G_+ = -\frac{1}{4} e^{-\phi/2-2A} \not{\partial} f_1 \Gamma_H, \quad G_- = -\frac{1}{4} e^{\phi/2-2B} \not{\partial} f_2 \Gamma_S, \quad (G_\pm)^* = \pm G_\pm \quad (\text{A.14})$$

and express the three form flux in terms of them:

$$\frac{1}{24} \mathcal{G} = G_+ + G_- \quad (\text{A.15})$$

Using all this information, we arrive at the final set of equations:

$$\not{\partial} \phi \epsilon^* - (G_+ + G_-) \epsilon = 0, \quad \not{\partial} \phi \epsilon - (G_+ - G_-) \epsilon^* = 0 \quad (\text{A.16})$$

$$(e^{-A} P_H + \not{\partial} A) \epsilon - i e^{-4C} \not{\partial} f_3 \Gamma_\Omega \epsilon + \frac{1}{2} (-3G_+ + G_-) \epsilon^* = 0 \quad (\text{A.17})$$

$$(-i e^{-B} P_S + \not{\partial} B) \epsilon - i e^{-4C} \not{\partial} f_3 \Gamma_\Omega \epsilon + \frac{1}{2} (G_+ - 3G_-) \epsilon^* = 0 \quad (\text{A.18})$$

$$(-i e^{-C} P_\Omega + \not{\partial} C) \epsilon + i e^{-4C} \not{\partial} f_3 \Gamma_\Omega \epsilon + \frac{1}{2} (G_+ + G_-) \epsilon^* = 0 \quad (\text{A.19})$$

$$\nabla_\mu \epsilon + i \frac{e^{-4C}}{2} \not{\partial} f_3 \gamma_\mu \Gamma_\Omega \epsilon + \frac{1}{96} (\gamma_\mu \mathcal{G} - 2\{\mathcal{G}, \gamma_\mu\}) \epsilon^* = 0. \quad (\text{A.20})$$

¹³We use the following sign conventions for $\Gamma_S, \Gamma_H, \Gamma_\Omega$. If Γ^0, Γ^9 are gamma matrices corresponding to AdS_2 factor, Γ^7, Γ^8 are matrices corresponding to S^2 and $\Gamma^3, \dots, \Gamma^6$ are the ones for S^4 , then we define $\Gamma_H = -\Gamma^0\Gamma^9, \Gamma_S = -i\Gamma^7\Gamma^8, \Gamma_\Omega = \Gamma^3\Gamma^4\Gamma^5\Gamma^6$. Also thorough this paper we use Γ to denote matrices with frame indices (which square to ± 1), and γ stands for the matrices with spacetime indices.

Notice that the second equation in (A.16) is just a complex conjugate of the first one, but for future reference it is convenient to keep them together. It would also be useful to write a hermitean conjugate of the last system:

$$\epsilon^T \not{\partial} \phi - \epsilon^\dagger (G_+ + G_-) = 0, \quad \epsilon^\dagger \not{\partial} \phi - \epsilon^T (G_+ - G_-) = 0 \quad (\text{A.21})$$

$$\epsilon^\dagger (e^{-A} P_H + \not{\partial} A) + ie^{-4C} \epsilon^\dagger \not{\partial} f_3 \Gamma_\Omega + \frac{1}{2} \epsilon^T (-3G_+ + G_-) = 0 \quad (\text{A.22})$$

$$\epsilon^\dagger (ie^{-B} P_S + \not{\partial} B) + ie^{-4C} \epsilon^\dagger \not{\partial} f_3 \Gamma_\Omega + \frac{1}{2} \epsilon^T (G_+ - 3G_-) = 0 \quad (\text{A.23})$$

$$\epsilon^\dagger (ie^{-C} P_\Omega + \not{\partial} C) - ie^{-4C} \epsilon^\dagger \not{\partial} f_3 \Gamma_\Omega + \frac{1}{2} \epsilon^T (G_+ + G_-) = 0 \quad (\text{A.24})$$

$$\nabla_\mu \epsilon^\dagger - i \frac{e^{-4C}}{2} \epsilon^\dagger \gamma_\mu \not{\partial} f_3 \Gamma_\Omega + \frac{1}{96} \epsilon^T (\mathcal{G}^\dagger \gamma_\mu - 2\{\mathcal{G}^\dagger, \gamma_\mu\}) = 0. \quad (\text{A.25})$$

To summarize, we showed that the problem of finding supersymmetric solution with metric and fluxes (A.6) reduces to solving the system (A.16)–(A.20). In the remaining part of this appendix we will simplify this system and show that its solutions can be parameterized in terms of one harmonic function.

A.2 Choosing coordinates and evaluating the metric

Before we start solving differential equations, it is useful to recall that metric (A.6) is invariant under reparameterizations of x_1, x_2 plane, and one can use this symmetry to choose a convenient coordinate system. We begin with adding equations (A.18), (A.19) and the second equation in (A.16):

$$[-ie^{-B} P_S - ie^{-C} P_\Omega + \not{\partial} (B + C + \phi)] \epsilon = 0. \quad (\text{A.26})$$

This is a projector which generically contains four gamma matrices, but by appropriate choice of coordinates and vielbein, we can express it in terms of three matrices. Namely we define a coordinate y by a relation $y = e^{B+C+\phi}$, then in two dimensions we can always choose another coordinate x to be orthogonal to y . Then metric becomes diagonal and we choose a convenient vielbein:

$$g_{ij} dx^i dx^j = g^2 (dy^2 + h^2 dx^2), \quad e_y^{\hat{y}} = g, \quad e_x^{\hat{x}} = gh. \quad (\text{A.27})$$

In this coordinate frame the relation (A.26) becomes

$$\left[-ie^{-B} P_S - ie^{-C} P_\Omega + \frac{1}{yg} \Gamma_y \right] \epsilon = 0. \quad (\text{A.28})$$

We already related the product of the warp factors of the spheres with the value of coordinate y , now it is convenient to parameterize their ratio by function G . Then the condition for the last equation to be a projector leads to an expression for g in terms of G :

$$e^{2B} = ye^{-\phi+G}, \quad e^{2C} = ye^{-\phi-G}, \quad g^2 = \frac{e^{-\phi}}{2y \cosh G}. \quad (\text{A.29})$$

The projector itself can also be expressed in terms of G :

$$\left[ie^{-G/2}P_S + ie^{G/2}P_\Omega - \sqrt{e^G + e^{-G}}\Gamma_y\right]\epsilon = 0 \quad (\text{A.30})$$

and this expression can be further simplified by introducing a rescaled spinor ϵ_1 :

$$\epsilon = e^{-\delta P_\Omega P_S}\epsilon_1, \quad (iP_\Omega - \Gamma_y)\epsilon_1 = 0, \quad \cos 2\delta = \frac{e^{G/2}}{\sqrt{e^G + e^{-G}}}. \quad (\text{A.31})$$

Once this projection is imposed, the equations (A.18), (A.19) and (A.16) become linearly dependent and we can disregard equation (A.18).

There is one more more combination of (A.16)–(A.19) which does not contain fluxes: adding (A.17) and (A.19) and subtracting (A.16), we find a projector¹⁴

$$\begin{aligned} (e^{-A}P_H - ie^{-C}P_\Omega + \not{\partial}(A + C - \phi))\epsilon &= 0 : \\ (e^{-A}P_H - ie^{-C}(c_\delta P_\Omega - s_\delta P_S) + \not{\partial}(A + C - \phi))\epsilon_1 &= 0. \end{aligned} \quad (\text{A.32})$$

We wrote this equation in terms of ϵ_1 because this spinor satisfies a very simple projection relation (A.31). In particular, acting on the last equation by $(\Gamma_y \pm iP_\Omega)$, we find two relations:

$$\begin{aligned} \left(e^{-A}P_H + ie^{-C}s_\delta P_S + \frac{1}{gh}\Gamma^x\partial_x(A + C - \phi)\right)\epsilon_1 &= 0, \\ \left(-e^{-C}c_\delta + \frac{1}{g}\partial_y(A + C - \phi)\right)\epsilon_1 &= 0. \end{aligned}$$

The first equation suggests that it is convenient to rescale a spinor one more time and to define a useful function F :

$$\epsilon_1 = e^{i\sigma P_S P_H}\epsilon_0, \quad \tanh 2\sigma = e^{-A}\sqrt{e^{2B} + e^{2C}}, \quad F = \sqrt{e^{2A} - e^{2B} - e^{2C}}. \quad (\text{A.33})$$

This leads to a simple expressions for the derivatives of $A + C - \phi$:

$$\partial_y(A + C - \phi) = \frac{e^G}{y(e^G + e^{-G})}, \quad \partial_x(A + C - \phi) = -\frac{\alpha h F e^{-A}}{y(e^G + e^{-G})} \quad (\text{A.34})$$

and to a simple projection relation

$$[iP_S - \alpha\Gamma_x]\epsilon_0 = 0. \quad (\text{A.35})$$

Here α is a parameter which is equal to plus or minus one, and we will fix its value later. As before, we conclude that the system (A.34), (A.35) can be viewed as a replacement for the equation (A.19).

¹⁴In this paper we encounter numerous trigonometric and hyperbolic functions of various arguments. To avoid writing formulas which are unnecessarily long, we adopt a shorthand notation:

$$s_x \equiv \sin x, \quad c_x \equiv \cos x, \quad sh_x \equiv \sinh x, \quad ch_x \equiv \cosh x$$

At this point the metric is still invariant under reparameterizations of x and it would be nice to find a convenient gauge. Under such reparameterizations, it is only function h which changes, so to fix the gauge we will need to know the y -dependence of h . The simplest way to address this question is to look at certain spinor bilinears and find the differential equations for them. Notice, that in principle to find a supersymmetric background it is sufficient to analyze all spinor bilinears, and this technique was very fruitful in the recent years [31]. In particular, in [4] it was used to find 1/2 BPS geometries with $SO(4) \times SO(4)$ symmetry which is analogous to the problem which we are considering. However in this paper we mostly work with spinors directly, and one of the reason for this is that one can construct many bilinears by placing matrices $P_H, P_S, P_\Omega, \Gamma_H, \Gamma_S, P_\Omega, \gamma_{1,2}$ between spinors and many of these bilinears turn out to be zero. Also some of the bilinears in the set are equal to others, so it appears that the exhaustive analysis of bilinears in the present case would be longer than a direct search for a solution of spinor equations. However we will now use some of the bilinears to determine the function h .

Since we want to find a differential equation for $e_x^{\hat{x}}$, it is natural to start from a vector bilinear which has only a component along x direction. The projection (A.31) shows that one such bilinear is¹⁵

$$\epsilon^\dagger \Gamma_\Omega \gamma_\mu P_\Omega \epsilon = \epsilon_1^\dagger \Gamma_\Omega \gamma_\mu P_\Omega \epsilon_1 : \quad \epsilon^\dagger \Gamma_\Omega \gamma_y P_\Omega \epsilon = 0, \quad \epsilon^\dagger \Gamma_\Omega \gamma_x P_\Omega \epsilon = e_x^{\hat{x}} \epsilon_1^\dagger \Gamma_\Omega \hat{\sigma}_2 \epsilon_1.$$

One-form constructed from this vector can be expressed in terms of a scalar bilinear:

$$\epsilon^\dagger \Gamma_\Omega \gamma_\mu P_\Omega \epsilon dx^\mu = e_x^{\hat{x}} e_y^{\hat{y}} e^{-\phi-B} \epsilon^\dagger \Gamma_\Omega \hat{\sigma}_2 \epsilon dx = h e^{-\phi-B} \epsilon^\dagger \Gamma_\Omega \hat{\sigma}_2 \epsilon dx. \quad (\text{A.36})$$

Knowing an exterior derivative of this vector as well as coordinate dependence of the scalar bilinear, we can extract a y -dependence of h . We begin with computation of the exterior derivative using (A.20):

$$\begin{aligned} \nabla_\mu (\epsilon^\dagger \Gamma_\Omega \gamma_\nu P_\Omega \epsilon) + \frac{i}{2} e^{-4C} \left(\epsilon^\dagger \Gamma_\Omega \gamma_\nu P_\Omega \not{\partial} f_3 \gamma_\mu \Gamma_\Omega \epsilon - hc \right) \\ - \frac{1}{4} \left[\epsilon^\dagger \Gamma_\Omega \gamma_\nu P_\Omega (\gamma_\mu (G_+ + G_-) + 2(G_+ + G_-) \gamma_\mu) \epsilon^* + hc \right] = 0. \end{aligned}$$

Using (A.16) to exclude $G_- \epsilon^*$ from this expression, and taking antisymmetric part in μ, ν indices, we find

$$\nabla_{[\mu} (\epsilon^\dagger \Gamma_\Omega \gamma_{\nu]} P_\Omega \epsilon) + \frac{1}{4} \left[\epsilon^\dagger \Gamma_\Omega P_\Omega (\gamma_{\nu\mu} (-\not{\partial} \phi \epsilon + 2G_+ \epsilon^*) + 2\gamma_{[\nu} (G_+ + G_-) \gamma_{\mu]} \epsilon^*) + hc \right] = 0.$$

Noticing that

$$\Gamma_x \Gamma_\lambda \Gamma_y - \Gamma_y \Gamma_\lambda \Gamma_x = 2\delta_\lambda^x (\Gamma_y - \Gamma_x) - \Gamma_\lambda \Gamma_{xy} - \Gamma_{xy} \Gamma_\lambda = 0$$

we simplify the equation above:

$$\nabla_{[\mu} (\epsilon^\dagger \Gamma_\Omega \gamma_{\nu]} P_\Omega \epsilon) - \frac{i}{4} \epsilon_{\mu\nu} \left[\epsilon^\dagger \Gamma_\Omega P_\Omega \hat{\sigma}_2 (-\not{\partial} \phi \epsilon + 2G_+ \epsilon^*) + hc \right] = 0. \quad (\text{A.37})$$

¹⁵We are using conventions $\Gamma_x \Gamma_y = i\hat{\sigma}_2$.

Now one can use the explicit form of G_+ along with relation $\Gamma_H = \Gamma_H^T$ to evaluate the transpose of the term involving field strength:

$$(\epsilon^\dagger \Gamma_\Omega P_\Omega \hat{\sigma}_2 G_+ \epsilon^*)^T = -\epsilon^\dagger \Gamma_\Omega P_\Omega \hat{\sigma}_2 G_+ \epsilon^* = 0. \quad (\text{A.38})$$

Then finally get an equation

$$\nabla_{[\mu}(\epsilon^\dagger \Gamma_\Omega \gamma_{\nu]} P_\Omega \epsilon) + \frac{i}{2} \epsilon_{\mu\nu} \epsilon^\dagger \Gamma_\Omega P_\Omega \hat{\sigma}_2 \not{\partial} \phi \epsilon = 0.$$

Writing this in terms of forms, and taking a coefficient in front of $dy \wedge dx$, we find:

$$\partial_y (h e^{-\phi-B} \epsilon^\dagger \Gamma_\Omega \hat{\sigma}_2 \epsilon) + h e^{-\phi-B} \epsilon^\dagger \Gamma_\Omega \hat{\sigma}_2 \epsilon \partial_y \phi = 0. \quad (\text{A.39})$$

To extract a y -dependence of h we need to know a functional form of the bilinear appearing in this relation. Starting from differential equations (A.20), (A.25) one can write an expression for the derivative of this bilinear, then using (A.16) to remove G_+ from the result, one arrives at

$$\nabla_\mu(\epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \epsilon) + i \frac{e^{-4C}}{2} \epsilon^\dagger \hat{\sigma}_2 [\not{\partial} f_3, \gamma_\mu] \epsilon - \left(\frac{1}{2} \epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \gamma_\mu G_- \epsilon^* - \frac{1}{4} \partial_\mu \phi \epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \epsilon + hc \right) = 0. \quad (\text{A.40})$$

The term involving three-form can be evaluated by looking at combination of (A.16) and (A.18):

$$\left[-ie^{-B} P_S + \not{\partial} \left(B + \frac{\phi}{2} \right) - ie^{-4C} \not{\partial} f_3 \Gamma_\Omega \right] \epsilon - G_- \epsilon^* = 0$$

and at the conjugate relation. This leads to equation

$$\frac{1}{2} \left(\epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \gamma_\mu G_- \epsilon^* + hc \right) = \epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \left[\partial_\mu \left(B + \frac{\phi}{2} \right) - \frac{i}{2} e^{-4C} [\gamma_\mu, \not{\partial} f_3] \Gamma_\Omega \right] \epsilon. \quad (\text{A.41})$$

Substituting this into the equation for the scalar bilinear, we find a very simple relation which can be solved in terms of one integration constant c_1 :

$$\nabla_\mu(\epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \epsilon) - \epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \epsilon \partial_\mu B = 0 : \quad \epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \epsilon = c_1 e^B \quad (\text{A.42})$$

We will show below that c_1 is not equal to zero (see equation (A.59)) and this fact will not rely on a particular value of h . It is only for presentational purposes that we postpone the derivation of (A.59) until the next subsection. Substituting (A.42) into (A.39) and dividing result by non-vanishing c_1 , we conclude that function h does not depend on y , so we can choose a gauge where $h = 1$.

To summarize, we fixed the diffeomorphism-invariance in the metric, and we shown that it can be written in terms of two independent warp factors and the dilaton:

$$ds^2 = e^{2A} dH_2^2 + e^{2B} d\Omega_2^2 + e^{2C} d\Omega_4^2 + \frac{e^{-2\phi}}{e^{2B} + e^{2C}} (dx^2 + dy^2) \quad (\text{A.43})$$

$$e^{2A} = ye^{H-\phi}, \quad e^{2B} = ye^{G-\phi}, \quad e^{2C} = ye^{-G-\phi}.$$

Notice that this result was already obtained in [14], but to be able to go further, we had to re-derive it in the standard notation. We also showed that the Killing spinor should satisfy four algebraic relations: (A.16), (A.17) and

$$[iP_S - \alpha\Gamma_x]\epsilon_0 = 0, \quad [iP_\Omega - \Gamma_y]\epsilon_0 = 0, \quad \epsilon = e^{-\delta P_\Omega P_S} e^{i\sigma P_S P_H} \epsilon_0 \quad (\text{A.44})$$

Along with differential equations (A.20) and (A.34), these relations give a complete system of equations which we solve in the next subsection. We conclude by rewriting the differential equations (A.34) in terms of G and H :

$$\frac{1}{2}\partial_y(H - G - 4\phi) = -\frac{e^{-G}}{y(e^G + e^{-G})}, \quad \frac{1}{2}\partial_x(H - G - 4\phi) = -\frac{\alpha F e^{-A}}{y(e^G + e^{-G})} \quad (\text{A.45})$$

A.3 Evaluating the fluxes

We begin with looking at the dilatino variation (A.20) and rewriting it in terms of ϵ_0 :

$$\not{\partial}\phi\epsilon_0^* - (e^{-2i\sigma P_S P_H} G_+ + c_{2\delta} e^{-2i\sigma P_S P_H} G_- + s_{2\delta} P_\Omega P_S G_-)\epsilon_0 = 0 \quad (\text{A.46})$$

We want to take various projections of this equation, and it seems convenient to write the matrices G_+ and G_- in terms of scalars:

$$G_+ \equiv G_{+,x}\gamma^x\Gamma_H + G_{+,y}\gamma^y\Gamma_H, \quad G_- \equiv G_{-,x}\gamma^x\Gamma_S + G_{-,y}\gamma^y\Gamma_S \quad (\text{A.47})$$

Let us act on (A.46) by $\gamma_x(1 + i\Gamma_y P_\Omega)$ then using the relation $P_\Omega P_S \Gamma_y \Gamma_x \epsilon_1 = \alpha \epsilon_1$, we arrive at the equation

$$\partial_x\phi\epsilon_0^* - (e^{-2i\sigma P_S P_H} G_{+,x}\Gamma_H + c_{2\delta} e^{-2i\sigma P_S P_H} G_{-,x}\Gamma_S + \alpha s_{2\delta} G_{-,y}\Gamma_S)\epsilon_0 = 0 \quad (\text{A.48})$$

Projecting this relation by $(1 \pm i\alpha\Gamma_x P_S)$, we find :

$$\begin{aligned} \partial_x\phi\epsilon_0^* - (ch_{2\sigma}G_{+,x}\Gamma_H - ic_{2\delta}sh_{2\sigma}P_S P_H G_{-,x}\Gamma_S)\epsilon_0 &= 0 \\ -i\Gamma_S P_S \Gamma_H P_H \epsilon_0 \quad sh_{2\sigma}G_{+,x} &= (c_{2\delta}ch_{2\sigma}G_{-,x} + \alpha s_{2\delta}G_{-,y})\epsilon_0 \end{aligned} \quad (\text{A.49})$$

Assuming the the three-form flux doesn't vanish, we conclude that there is an additional projection relation:

$$i\Gamma_S P_S \Gamma_H P_H \epsilon_0 = \beta\epsilon_0 \quad (\text{A.50})$$

where $\beta = \pm 1$. Then equations (A.49) can be rewritten as

$$c_{2\delta}ch_{2\sigma}G_{-,x} + \alpha s_{2\delta}G_{-,y} + \beta sh_{2\sigma}G_{+,x} = 0 \quad (\text{A.51})$$

$$\partial_x\phi\Gamma_H\epsilon_0^* - (ch_{2\sigma}G_{+,x} + \beta c_{2\delta}sh_{2\sigma}G_{-,x})\epsilon_0 = 0 \quad (\text{A.52})$$

Since all coefficients in the last equation are real, for solutions with nontrivial dilaton there is one more restriction on ϵ_0 :

$$\Gamma_H\epsilon_0^* = a\epsilon_0, \quad a = \pm 1. \quad (\text{A.53})$$

Similarly, acting on (A.46) by $\gamma_y(1 - i\Gamma_y P_\Omega)$, we find

$$c_{2\delta} ch_{2\sigma} G_{-,y} - \alpha s_{2\delta} G_{-,x} + \beta sh_{2\sigma} G_{+,y} = 0 \quad (\text{A.54})$$

$$\partial_y \phi \Gamma_H \epsilon_0^* - (ch_{2\sigma} G_{+,y} + \beta c_{2\delta} sh_{2\sigma} G_{-,y}) \epsilon_0 = 0 \quad (\text{A.55})$$

Notice that if dilaton is equal to constant, then we get a homogeneous system of equations for four components of flux, and the determinant of the appropriate matrix is equal to $ch_{2\sigma}^2 - \frac{1}{2} sh_{2\sigma}^2 s_{4\delta} > 0$, so if we want a solution with nontrivial three-form flux, the dilaton should not vanish and projection (A.53) should be enforced. For a vanishing three-form flux, the spinor can be chosen to be real, but we can choose a modified "reality condition" (A.53) as well. Substituting the expressions for $G_{\pm,x}$ and $G_{\pm,y}$ in (A.51)–(A.55) and rewriting the result in terms of differential forms, we arrive at two equations:

$$-\beta sh_{2\sigma} e^{-\phi/2-2A} df_1 = e^{\phi/2-2B} [c_{2\delta} ch_{2\sigma} df_2 + \alpha s_{2\delta} * df_2] \quad (\text{A.56})$$

$$ad\phi = -\frac{1}{4} \left[ch_{2\sigma} e^{-\phi/2-2A} df_1 + \beta e^{\phi/2-2B} c_{2\delta} sh_{2\sigma} df_2 \right] \quad (\text{A.57})$$

Here and below the star represents a Hodge duality in two dimensions with a sign convention: $*dy = dx$. The last two relations can be viewed as equations for df_1 and df_2 and straightforward algebraic manipulations lead to the solution of this system:

$$\begin{aligned} df_1 &= -\frac{4ae^{2A+\phi/2}}{e^{2A} - e^{2B}} [e^A F d\phi - \alpha e^{B+C} * d\phi] \\ df_2 &= 4a\beta \frac{e^{-\phi/2+2B}}{e^{2A} - e^{2B}} [e^B F d\phi - \alpha e^{A+C} * d\phi] \end{aligned} \quad (\text{A.58})$$

Let us pause for a moment and collect all projection relations which have been imposed on ϵ_0 so far. We have (A.44), (A.50), (A.53) and the standard projector with Γ_{11} , and these five projectors commute with each other. One can also check that these projectors are independent (for example, using the explicit basis (A.12)), so they reduce a dimension of a spinor by a factor of $2^5 = 32$. Notice that in the basis (A.12) we have a 16-component complex spinor, so the projections imply that it can be parameterized in terms of one real function¹⁶. This explains why we chose to work with spinor directly rather than to write down all bilinears following [4]: to determine the spinor completely we only need one real bilinear out of a large set of expressions. In fact we already encountered a useful bilinear in (A.42), now we will take a closer look at it.

First we want to show that c_1 is a non-vanishing constant. To this end we will use various projectors to express the bilinear (A.42) in terms of ϵ_0 :

$$\epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \epsilon = c_{2\delta} \epsilon_0^\dagger \hat{\sigma}_2 \Gamma_\Omega (i sh_{2\sigma} P_S P_H) \epsilon_0 = \beta c_{2\delta} sh_{2\sigma} \epsilon_0^\dagger \hat{\sigma}_2 \Gamma_\Omega \Gamma_S \Gamma_H \epsilon_0 = -\beta c_{2\delta} sh_{2\sigma} \epsilon_0^\dagger \epsilon_0 \quad (\text{A.59})$$

Here we used the definition of Γ_{11} as well as projection which it imposes:

$$\Gamma_{11} = -i\Gamma_x \Gamma_y \Gamma_\Omega \Gamma_S \Gamma_H = \hat{\sigma}_2 \Gamma_\Omega \Gamma_S \Gamma_H, \quad \Gamma_{11} \epsilon = -\epsilon. \quad (\text{A.60})$$

¹⁶Of course, the spinor in type IIB supergravity has 32 complex components (before the Γ_{11} projection is imposed), but we suppressed the directions along the spheres and AdS. So starting with complex spinor which has one real component in our notation, we produce an object which has $2 \times 2 \times 4 = 16$ real components. This is expected since we are looking at states which preserve 1/2 of supersymmetries.

Equation (A.59) implies that unless the Killing spinor ϵ_0 is identically equal to zero, the bilinear $\epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \epsilon$ does not vanish, which proves that coefficient c_1 in (A.42) is not equal to zero¹⁷, then we can rescale a spinor ϵ_0 to set $c_1 = -\beta$. Now equation (A.40) can be viewed as a differential equation for B , and since one bilinear determined the Killing spinor completely, the relation (A.40) along with projectors that we discussed is equivalent to (A.20). Substituting the value of G_- into (A.40), we find a simple differential equation:

$$\partial_\mu(-\beta e^B) + e^{-4C} \epsilon^\dagger \epsilon \epsilon_{\mu\nu} \partial_\nu f_3 - \frac{\beta}{2} e^B \partial_\mu \phi + \frac{1}{8} e^{-2B+\phi/2} \partial_\mu f_2 \left[\epsilon^\dagger \hat{\sigma}_2 \Gamma_\Omega \Gamma_S \epsilon^* + cc \right] = 0. \quad (\text{A.61})$$

Using various projectors, we evaluate the bilinears that appear in this expression:

$$\epsilon^\dagger \epsilon = ch_{2\sigma} \epsilon_0^\dagger \epsilon_0 = \frac{e^A}{F} \frac{e^B}{c_{2\delta} sh_{2\sigma}} = \frac{e^A}{F} F = e^A \quad (\text{A.62})$$

$$\epsilon^T \hat{\sigma}_2 \Gamma_\Omega \Gamma_S \epsilon = -\epsilon_0^T \Gamma_H \epsilon_0 = -a \epsilon_0^\dagger \epsilon_0 = -aF \quad (\text{A.63})$$

Substituting this into (A.61) and rewriting the result in terms of forms, we find the expression for the five-form flux:

$$\beta e^{-4C+A} * df_3 = e^B d \left(B + \frac{\phi}{2} \right) + \frac{a\beta}{4} F e^{-2B+\phi/2} df_2. \quad (\text{A.64})$$

This equation replaces (A.20). For future reference we write an alternative form of the last equation, which can be obtained by combining it with (A.34) and (A.58):

$$\beta e^{-4C+B} * df_3 = e^A d \left(A - \frac{\phi}{2} \right) + \frac{a}{4} F e^{-2A-\phi/2} df_1 \quad (\text{A.65})$$

At this point the complete system of bosonic equations is given by (A.34), (A.58), (A.65), and in addition we should keep one of the three equations (A.17)–(A.19). Let us look at (A.18) and rewrite it in terms of ϵ_0 :

$$\left[-ie^{-B} P_S + e^{i\sigma P_S P_H} e^{-2\delta P_\Omega P_S} e^{i\sigma P_S P_H} \not{\partial} \left(B + \frac{\phi}{2} \right) - ie^{-4C} e^{2i\sigma P_S P_H} \not{\partial} f_3 \Gamma_\Omega \right] \epsilon_0 - G_- \epsilon_0^* = 0. \quad (\text{A.66})$$

Here we also used the dilatino equation to eliminate G_+ . To proceed it is useful to combine the projection relations for ϵ_0 to construct one more projector:

$$S = \alpha \Gamma_x \Gamma_y P_S P_\Omega : \quad S \epsilon_1 = \epsilon_1, \quad S \epsilon_1^* = \epsilon_1^* \quad (\text{A.67})$$

We can now decompose (A.66) into two equations by applying $1 \pm S$ to it. It turns out that after acting by $1 + S$ on (A.66), we get an equation which is equivalent to (A.64), however acting by $1 - S$ we find a new relation:

$$\begin{aligned} & \left[-ie^{-B} P_S + (c_{2\delta} ch_{2\sigma} - P_\Omega P_S s_{2\delta}) \not{\partial} \left(B + \frac{\phi}{2} \right) + e^{-4C} sh_{2\sigma} P_S P_H \not{\partial} f_3 \Gamma_\Omega \right] \epsilon_0 = 0 \\ & \left[-ie^{-B} \alpha \hat{\sigma}_2 \Gamma_y + (c_{2\delta} ch_{2\sigma} + i\alpha \hat{\sigma}_2 s_{2\delta}) \not{\partial} \left(B + \frac{\phi}{2} \right) + i\beta \hat{\sigma}_2 e^{-4C} sh_{2\sigma} \not{\partial} f_3 \right] \epsilon_0 = 0. \end{aligned}$$

¹⁷Although in this subsection we already took $h = 1$, one can show that all projection relations remain the same for an arbitrary h , so to arrive at relation (A.59) one does not rely on equation (A.42) (otherwise the logic would be circular). We chose to write equations in this order only to avoid unnecessary complications.

Noticing that the first term can be expressed in terms of derivative as

$$-ie^{-B}\alpha\hat{\sigma}_2\Gamma_y = -i\alpha\hat{\sigma}_2\frac{e^{-B-\phi}}{\sqrt{e^{2B}+e^{2C}}}\partial y = -i\alpha\hat{\sigma}_2s_{2\delta}\partial\log y, \quad (\text{A.68})$$

we find the last projection relation:

$$\left[ic_{2\delta}ch_{2\sigma}\hat{\sigma}_2\partial\left(B+\frac{\phi}{2}\right) - \alpha s_{2\delta}\partial\left(B+\frac{\phi}{2}-\log y\right) + \beta e^{-4C}sh_{2\sigma}\partial f_3\right]\epsilon_0 = 0. \quad (\text{A.69})$$

As we mentioned before, at this point ϵ_0 is essentially a one-component spinor, so we cannot impose any more restrictions on it. This implies that in (A.69) the coefficients in front of Γ_x and Γ_y have to vanish separately. An alternative way of seeing this to act on (A.69) by $(1 - i\alpha\Gamma_x P_S)$ and to use the projector (A.35). Thus we end up with equation

$$\beta e^{-4C} * df_3 = \frac{e^{A+B}}{e^{2B}+e^{2C}} d\left(B+\frac{\phi}{2}\right) + \alpha \frac{F e^C}{e^{2B}+e^{2C}} * d\left(B+\frac{\phi}{2}-\log y\right). \quad (\text{A.70})$$

We can exclude five-form flux f_3 from the last equation by combining it with (A.64), then using (A.45) we find the relation

$$-\frac{e^{B+C}dG}{e^{2B}+e^{2C}} + \frac{\alpha e^A}{2F} * d\log\frac{e^G+e^{-G}}{e^H} - \frac{a\alpha}{2}e^{-\phi/2-2A} * df_1 = 0. \quad (\text{A.71})$$

To summarize, we have shown that the system of five differential equations (A.16)–(A.20) can be rewritten as four projectors (A.44), (A.50), (A.53), and five differential relations (A.45), (A.58), (A.65), (A.71) and these two descriptions are equivalent. For future reference, in the next subsection we collect all equations in one place.

A.4 Summarizing supergravity solution

In this long appendix we analyzed the SUSY variations of type IIB supergravity on a manifold with $AdS_2 \times S^2 \times S^4$ factors. Let us now collect the results. We showed that one can always choose a coordinate system so the the metric and fluxes have a form

$$ds^2 = e^{2A}dH_2^2 + e^{2B}d\Omega_2^2 + e^{2C}d\Omega_4^2 + \frac{e^{-2\phi}}{e^{2B}+e^{2C}}(dx^2 + dy^2) \quad (\text{A.72})$$

$$F_5 = df_3 \wedge d\Omega_4 + *_{10}(df_3 \wedge d\Omega_4), \quad H_3 = df_1 \wedge dH_2, \quad F_3 = df_2 \wedge d\Omega_2 \quad (\text{A.73})$$

$$e^{2A} = ye^{H-\phi}, \quad e^{2B} = ye^{G-\phi}, \quad e^{2C} = ye^{-G-\phi}, \quad F = \sqrt{e^{2A} - e^{2B} - e^{2C}} \quad (\text{A.74})$$

The geometry is supersymmetric if and only if these fields satisfy the following differential relations

$$df_1 = -\frac{4ae^{2A+\phi/2}}{e^{2A}-e^{2B}} [e^A F d\phi - \alpha e^{B+C} * d\phi], \quad (\text{A.75})$$

$$df_2 = \frac{4a\beta e^{2B-\phi/2}}{e^{2A}-e^{2B}} [e^B F d\phi - \alpha e^{A+C} * d\phi] \quad (\text{A.76})$$

$$\beta e^B e^{-4C} * df_3 = e^A d\left(A - \frac{\phi}{2}\right) + \frac{a}{4} F e^{-\phi/2-2A} df_1 \quad (\text{A.77})$$

$$d(H - G - 4\phi) = -\frac{2}{y(e^{2B}+e^{2C})}(e^{2C} dy + \alpha F e^{B+C-A} dx) \quad (\text{A.78})$$

$$\alpha * d\arctan e^G + \frac{1}{2}d\log\frac{e^A - F}{e^A + F} - \frac{a}{2}e^{-\phi/2-2A} df_1 = 0. \quad (\text{A.79})$$

In this coordinate system we expressed the Killing spinor ϵ in terms of a reduced spinor ϵ_0 which effectively has one real component due to projections imposed on it:¹⁸

$$\epsilon = e^{-\delta P_\Omega P_S} e^{i\sigma P_S P_H} \epsilon_0, \quad \tan 2\delta = e^{-G}, \quad \tanh 2\sigma = e^{-A} \sqrt{e^{2B} + e^{2C}} \quad (\text{A.80})$$

$$\epsilon_0 = -\Gamma_{11} \epsilon_0 = i\Gamma_y P_\Omega \epsilon_0 = i\alpha \Gamma_x P_S \epsilon_0 = a \Gamma_H \epsilon_0^* = i\beta \Gamma_S P_S \Gamma_H P_H \epsilon_0 \quad (\text{A.81})$$

We also determined the spinor ϵ_0 (up to overall normalization) by computing its bilinear:

$$\epsilon_0^\dagger \epsilon_0 = \sqrt{e^{2A} - e^{2B} - e^{2C}} \quad (\text{A.82})$$

Each of the constants a, α, β can be equal to either plus or minus one, and so far we have not fixed their signs. To avoid unnecessary complications, we will fix these projections in the main body of the paper by taking $a = \alpha = \beta = 1$. This choice does not make the situation less general, moreover a, α, β can be recovered easily by noticing that the differential relations written in this subsection remain invariant if we flip signs of all elements in any of the following sets:

$$(a, f_1, f_2), \quad (\alpha, x), \quad (\beta, f_2, f_3) \quad (\text{A.83})$$

Here x is one of the coordinates which so far was defined as being orthogonal to y and thus its sign was not fixed up to this point.

We conclude this summary by writing two useful relations. By combining (A.65) and (A.64) we arrive at the following equation:

$$\frac{1}{2} d[e^{2A} - e^{2B}] - \frac{1}{2} (e^{2A} + e^{2B}) d\phi + \frac{a}{2} F e^A e^{-\phi/2-2A} df_1 + F^2 d\phi = 0 \quad (\text{A.84})$$

and (A.64), (A.45), (A.71) can be combined into

$$\alpha e^{B+C} * d \left(C - \frac{\phi}{2} \right) - \frac{a}{4} e^{2B} e^{-\phi/2-2A} df_1 - F e^A d \left(A - \frac{\phi}{2} \right) - \frac{a}{4} F^2 e^{-\phi/2-2A} df_1 = 0. \quad (\text{A.85})$$

B. Constructing spinors on the sphere

While deriving the supersymmetry variations, we encountered spinors on unit spheres in even dimensions and in this appendix we summarize a construction of such spinors. First we recall that on even-dimensional sphere there are two types of Killing spinors, each class satisfies one of the equations [32]:

$$\nabla_m \epsilon_\pm^{(1)} = \pm \frac{i}{2} \gamma_m \epsilon_\pm^{(1)}, \quad (\text{B.1})$$

$$\nabla_m \epsilon_\pm^{(2)} = \pm \frac{1}{2} \gamma_m \epsilon_\pm^{(2)} \quad (\text{B.2})$$

Here γ is a hermitean chirality matrix. The situation is different for an odd dimensional sphere which has only one class of Killing spinors (which will be denoted $\hat{\epsilon}$ in this appendix):

$$\nabla_m \hat{\epsilon}_\pm = \pm \frac{i}{2} \gamma_m \hat{\epsilon}_\pm. \quad (\text{B.3})$$

¹⁸Nevertheless the solution is 1/2 BPS: see footnote 16.

Once we have a spinor on an odd-dimensional sphere S^n , it can be easily embedded in higher dimensions by using $n + p$ split of gamma matrices:

$$\Gamma_m = \sigma \otimes \gamma_m, \quad \Gamma_\mu = \sigma_\mu \otimes 1, \quad \{\sigma, \sigma_\mu\} = 0, \quad \mu = 1, \dots, p. \quad (\text{B.4})$$

Unfortunately such simple decomposition does not work if n is even and the goal of this appendix is to describe a construction of Killing spinors in that case. To do so we find it useful to study embeddings $S^n \rightarrow S^{n+1}$ and extract some general lessons from this construction.

We begin with reviewing this construction for odd n . Writing the metric on S^{n+1} as

$$d\Omega_{n+1}^2 = d\theta^2 + s_\theta^2 d\Omega_n^2, \quad (\text{B.5})$$

we find the derivatives along Ω_n and θ directions:

$$\nabla_i \epsilon = \frac{i}{2} \gamma_i \epsilon - \frac{1}{2} c_\theta \sigma_\theta \sigma \tilde{\gamma}_i \epsilon = \frac{i}{2s_\theta} \gamma_i (\sigma - ic_\theta \sigma_\theta) \epsilon, \quad \nabla_\theta \epsilon = \partial_\theta \epsilon. \quad (\text{B.6})$$

To reproduce the equation on the sphere we need to impose a projection:

$$0 = (\sigma - ic_\theta \sigma_\theta + \lambda s_\theta \sigma \sigma_\theta) \epsilon = e^{i\lambda \sigma \frac{\theta}{2}} (\sigma - i\sigma_\theta) e^{-i\lambda \sigma \frac{\theta}{2}} \epsilon. \quad (\text{B.7})$$

With this projection we reproduce the ‘‘chiral’’ relation on S^{n+1} :

$$\nabla_i \epsilon = -\frac{i\lambda}{2} \gamma_i \Gamma_\theta \epsilon, \quad \nabla_\theta \epsilon = \frac{i\lambda}{2} \Gamma_\theta \epsilon = -\frac{i\lambda}{2} \Gamma_\theta \Gamma_\theta \epsilon, \quad \epsilon = e^{i\lambda \Gamma_\theta \frac{\theta}{2}} \epsilon_0. \quad (\text{B.8})$$

Notice that by starting from $\hat{\epsilon}_+$ one can reproduce relations for both $\epsilon_+^{(2)}$ and $\epsilon_-^{(2)}$. Alternatively we could have started from $\hat{\epsilon}_-$ and produce both spinors on S^{n+1} . Such ambiguity is related to the fact that dimension of the spinor grows as we move from even to odd dimension. On the other hand, if we start from an even-dimensional sphere and add one more dimension, then the size of the spinor does not change and one needs to use both $\epsilon_+^{(2)}$ and $\epsilon_-^{(2)}$ to construct a spinor in higher dimension. We will now describe the relevant procedure.

We begin with ‘‘chiral’’ relation on even-dimensional sphere S^n and write the derivative on S^{n+1} :

$$\tilde{\nabla}_i \epsilon_a = \frac{a}{2} \tilde{\gamma}_i \Gamma_\theta \epsilon_a : \quad \nabla_i \epsilon_a = \frac{a}{2s_\theta} \gamma_i \Gamma_\theta \epsilon_a - \frac{c_\theta}{2s_\theta} \Gamma_\theta \gamma_i \epsilon_a = \frac{1}{2s_\theta} \gamma_i \Gamma_\theta (a + c_\theta) \epsilon_a. \quad (\text{B.9})$$

Clearly this does not reduce to relation (B.3), but we can define a new spinor

$$\hat{\epsilon}_+ \equiv \epsilon_+ + \epsilon_- \quad (\text{B.10})$$

then projection (B.2) for $\hat{\epsilon}_+$ leads to the equation

$$(1 + c_\theta - is_\theta \Gamma_\theta) \epsilon_+ = -(-1 + c_\theta - is_\theta \Gamma_\theta) \epsilon_- : \quad (1 + e^{-i\theta \Gamma_\theta}) \epsilon_+ = (1 - e^{-i\theta \Gamma_\theta}) \epsilon_-.$$

This equation and relation (B.3) along θ direction can be solved simultaneously by expressing ϵ_+ and ϵ_- in terms of θ -independent spinor ϵ_0 :

$$\epsilon_+ = i \sin \frac{\theta \Gamma_\theta}{2} \epsilon_0, \quad \epsilon_- = \cos \frac{\theta \Gamma_\theta}{2} \epsilon_0, \quad \hat{\epsilon}_+ = \exp \left(\frac{i\theta \Gamma_\theta}{2} \right) \epsilon_0 \equiv \epsilon_- + \Gamma_\theta \tilde{\epsilon}_+. \quad (\text{B.11})$$

This concludes the construction of a spinor on odd-dimensional sphere in terms of a "chiral" spinor on the even-dimensional one, but to learn a more general lesson about the spinors it is convenient to rewrite the above relation in a slightly different form.

Let us go back to the equation for ϵ_{\pm} and combine them into a single relation:

$$\tilde{\nabla}_i(\epsilon_- + \Gamma_{\theta}\tilde{\epsilon}_+) = \frac{1}{2}\tilde{\gamma}_i\Gamma_{\theta}(-\epsilon_- + \Gamma_{\theta}\tilde{\epsilon}_+). \quad (\text{B.12})$$

This equation can be rewritten in terms of spinor ϵ_0 and two matrices $\Gamma_{\pm} = \frac{1}{2}(1 \pm \Gamma_{\theta})$:

$$\tilde{\nabla}_i(e^{i\theta/2}\Gamma_+ + e^{-i\theta/2}\Gamma_-)\epsilon_0 = -\frac{1}{2}\tilde{\gamma}_i\Gamma_{\theta}(e^{-i\theta/2}\Gamma_+ + e^{i\theta/2}\Gamma_-)\epsilon_0 = \frac{1}{2}\tilde{\gamma}_i(-e^{-i\theta/2}\Gamma_+ + e^{i\theta/2}\Gamma_-)\epsilon_0.$$

In other words, we have two relations

$$\tilde{\nabla}_i\Gamma_+\epsilon_0 = \frac{1}{2}\tilde{\gamma}_i\Gamma_-\epsilon_0, \quad \tilde{\nabla}_i\Gamma_-\epsilon_0 = -\frac{1}{2}\tilde{\gamma}_i\Gamma_+\epsilon_0. \quad (\text{B.13})$$

To summarize, we found that a spinor on an odd-dimensional sphere can be decomposed as

$$\hat{\epsilon}_+ = \frac{1}{2}(e^{i\theta/2}\Gamma_+\epsilon_0 + e^{-i\theta/2}\Gamma_-\epsilon_0) \quad (\text{B.14})$$

and spinor ϵ_0 satisfies the equations (B.13). If we want to make the symmetries of S^n explicit, it is convenient to choose a basis of gamma matrices which has a form (B.4). Strictly speaking, this cannot be done for even n since the number of components of a spinor does not change as we go from even to odd dimension, however we can first double the size of the Killing spinor and then impose a projection. In the case of $S^n \rightarrow S^{n+1}$ lift we can choose a basis of gamma matrices

$$\Gamma_i = \hat{\sigma}_1 \otimes \tilde{\Gamma}_i, \quad \Gamma_{\theta} = \hat{\sigma}_3 \otimes 1 \quad (\text{B.15})$$

and then require that Killing spinor satisfies a constraint which involves a chirality operator γ :

$$\hat{\sigma}_3 \otimes \gamma \cdot \epsilon = \epsilon \quad (\text{B.16})$$

In particular, doubling the size of a spinor ϵ_0 and imposing a constraint, one rewrites equations (B.3) in terms of chirality matrix on S^n :

$$\tilde{\nabla}_i(1 + \gamma)\epsilon_0 = \frac{1}{2}\tilde{\gamma}_i(1 - \gamma)\epsilon_0, \quad \tilde{\nabla}_i(1 - \gamma)\epsilon_0 = -\frac{1}{2}\tilde{\gamma}_i(1 + \gamma)\epsilon_0. \quad (\text{B.17})$$

While we used the lift $S^n \rightarrow S^{n+1}$ to motivate this relation, the result can be applied to a general embedding of even dimensional spheres into higher dimensional spaces. One first introduces a basis of gamma matrices:

$$\Gamma_m = \sigma \otimes \tilde{\Gamma}_m, \quad \Gamma_{\mu} = \sigma_{\mu} \otimes 1, \quad \{\sigma, \sigma_{\mu}\} = 0, \quad \mu = 1, \dots, p. \quad (\text{B.18})$$

The spinors are constrained by projection involving chirality operator γ on the sphere:

$$\sigma \otimes \gamma \cdot \epsilon = \epsilon. \quad (\text{B.19})$$

Finally, the equations along the sphere directions are given by (B.17). To use these equations, it is convenient to rewrite them as

$$\tilde{\nabla}_m \epsilon = -\frac{i}{2}(\sigma \tilde{\gamma}_m) P \epsilon \tag{B.20}$$

and describe the properties of matrix P . First of all, since it contains a factor of σ , it anticommutes with all Γ_μ . We will also need the commutation relation for P and γ , and the simplest way to find it is to choose an explicit representation

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I \tag{B.21}$$

where I is a unit matrix involving irrelevant components of the spinor. Then equations (B.17) imply that in this representation matrix P has a form:

$$P = \sigma \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes I. \tag{B.22}$$

This shows that P is hermitean matrix which anticommutes with γ .

Let us now summarize the construction of Killing spinors on an even dimensional sphere. One defines a basis of gamma matrices (B.18) and imposes a projection (B.19) on a spinor. Then the Killing spinor satisfies an equation (B.20) with hermitean matrix P which anticommutes with γ_μ and with chirality operator γ .

Finally, we make a brief comment about AdS space. All formulas for the spheres can be rewritten for this case using a simple analytic continuation, in particular in equation (B.20) a prefactor $-i/2$ should be replaced by $1/2$.

C. Perturbative solution at large distances

In section 5 we outlined a procedure for constructing a solution as a perturbative series around $AdS_5 \times S^5$. For all solutions which asymptote to $AdS_5 \times S^5$ (i.e. for all functions Φ such that $\partial_y \Phi = 0$ in a finite region of $y = 0$ line) we expect this series to converge for large values of $x^2 + y^2$, then the entire solution can be constructed as an analytic continuation of the series. Some intermediate steps were missing in section 5 and here we will fill the gaps.

Let us consider a first order corrections to the fields, i.e. we take $m = 1$ in equations (5.1). Then looking at definition of Ψ_1, Ψ_2 , we find

$$d\Psi_2^{(1)} + *d\Psi_1^{(1)} = d(x\phi^{(1)}) - *d(y\phi^{(1)}). \tag{C.1}$$

Let us now expand the functions which enter equation (3.16):

$$\arctan e^G = \arctan e^{G_0} + \frac{\epsilon g^{(1)}}{e^{G_0} + e^{-G_0}} = \arctan e^{G_0} + \frac{\epsilon s sh g^{(1)}}{s^2 + sh^2}$$

$$F = \sqrt{c^2 + \epsilon[ch^2 h^{(1)} - sh^2 g^{(1)} + s^2 g^{(1)} - c^2 \phi^{(1)}]}$$

$$\begin{aligned}
&= c \left[1 - \frac{1}{2}\phi^{(1)} + \frac{\epsilon}{2c^2}(h^{(1)} + (h^{(1)} - g^{(1)})sh^2 + g^{(1)}s^2) \right] \equiv c(1 + \epsilon f^{(1)}) \\
e^A &= ch \left[1 - \epsilon \frac{\phi^{(1)}}{2} + \epsilon \frac{h^{(1)}}{2} \right] \\
\frac{e^A - F}{e^A + F} &= \frac{ch - c}{ch + c} \left[1 + \epsilon \frac{2Fe^A}{e^{2A} - F^2}(a^{(1)} - f^{(1)}) \right] \\
\left[\log \frac{e^A - F}{e^A + F} \right]^{(1)} &= \frac{2Fe^A}{e^{2A} - F^2}(a^{(1)} - f^{(1)}) = \frac{x}{sh^2 + s^2} \left[h^{(1)} - \frac{h^{(1)}ch^2}{c^2} + g^{(1)}\frac{sh^2 - s^2}{c^2} \right].
\end{aligned}$$

Substituting this into the equations (3.16), we arrive at the expressions:

$$\begin{aligned}
\frac{g^{(1)}}{s^2 + sh^2} - 2\phi^{(1)} &= \frac{\partial_y \Phi^{(1)}}{y}, \\
\frac{1}{2(sh^2 + s^2)} \left[h^{(1)} - \frac{h^{(1)}ch^2}{c^2} + g^{(1)}\frac{sh^2 - s^2}{c^2} \right] + 2\phi^{(1)} &= \frac{\partial_x \Phi^{(1)}}{x}. \tag{C.2}
\end{aligned}$$

It is useful to rewrite the last equation in a different form:

$$\begin{aligned}
\frac{\partial_x \Phi^{(1)}}{x} &= \frac{1}{2(sh^2 + s^2)} \left[(g^{(1)} - h^{(1)})\frac{sh^2 + s^2}{c^2} - 2g^{(1)}\frac{s^2}{c^2} \right] + 2\phi^{(1)} \\
&= -\frac{1}{2c^2}(h^{(1)} - g^{(1)} - 4\phi^{(1)}) + \frac{1}{2(sh^2 + s^2)} \left[-2g^{(1)}\frac{s^2}{c^2} \right] - 2\phi^{(1)}\frac{s^2}{c^2} \\
&= -\frac{1}{2c^2}(h^{(1)} - g^{(1)} - 4\phi^{(1)}) - \frac{s^2}{c^2}\frac{\partial_y \Phi^{(1)}}{y} - \frac{4s^2}{c^2}\phi^{(1)}. \tag{C.3}
\end{aligned}$$

At this point we can express everything in terms of $\Phi^{(1)}$ and $g^{(1)}$:

$$\begin{aligned}
2\phi^{(1)} &= \frac{g^{(1)}}{s^2 + sh^2} - \frac{\partial_y \Phi^{(1)}}{y}, \tag{C.4} \\
h^{(1)} - g^{(1)} - 4\phi^{(1)} &= - \left\{ 2s^2\frac{\partial_y \Phi^{(1)}}{y} + 8s^2\phi^{(1)} + 2c^2\frac{\partial_x \Phi^{(1)}}{x} \right\}.
\end{aligned}$$

To determine $g^{(1)}$ in terms of $\Phi^{(1)}$ we should use the equation

$$\partial_y(h^{(1)} - g^{(1)} - 4\phi^{(1)}) = \frac{4y}{s^2 + sh^2}\frac{g^{(1)}}{s^2 + sh^2} \tag{C.5}$$

which is a counterpart of (5.7) for $m = 1$. Evaluating the left-hand side of this relation:

$$\partial_y(h^{(1)} - g^{(1)} - 4\phi^{(1)}) = -4\partial_y \left(\frac{s^2 g^{(1)}}{s^2 + sh^2} \right) + 2\partial_y \left[s^2\frac{\partial_y \Phi^{(1)}}{y} - c^2\frac{\partial_x \Phi^{(1)}}{x} \right],$$

we arrive at equation (5.8) for $m = 1$. It is clear that the same set of equations which we derived now for $m = 1$ would hold for any m , the only difference would be in the source terms $\Phi^{(m)}$ and $\Psi_y^{(m)}$.

We were able to derive an equation for the perturbation in a closed form in part due to a miraculous relation (C.1). We recall that in general to find the potentials Ψ_1 and Ψ_2 in terms of the metric and the dilaton one needs to solve differential equations, but in the leading order the relation $dx = *dy$ led to algebraic expression for the potentials. One can hope that similar algebraic relation persists to higher orders as well. In the remaining part of this appendix we will analyze this question for the second order in perturbation and we conclude that there is no relation of the type (C.1). Another purpose to present these calculations here is to provide expressions for the first order correction to the metric in a form which is more explicit than (5.5), (5.8).

Let us look at the second correction to (3.13):

$$d\Psi_2^{(2)} + *d\Psi_1^{(2)} = d(x\phi^{(2)}) - *d(y\phi^{(2)}) + \left(\frac{e^A F}{e^{2A} - e^{2B}}\right)^{(1)} d\phi^{(1)} - y \left(\frac{e^{-\phi}}{e^{2A} - e^{2B}}\right)^{(1)} *d\phi^{(1)}. \quad (C.6)$$

We want to see whether the second line of this relation admits a simple decomposition similar to the one in the first line. To answer this question we have to evaluate the second line and try to guess an appropriate decomposition. While we can construct a complete solution in the linear order starting from any harmonic function Φ , such solutions look quite complicated due to presence of hyperbolic functions. One can first try to address the question in the "Poincare patch" of AdS space, i.e. in the region where hyperbolic functions can be replaced by the exponents. In this region the solution simplifies dramatically and we can write explicit expressions for all metric components. Notice that our perturbative expansion should work only at large values of $x^2 + y^2$, i.e. precisely in the regime of validity of the "Poincare patch." However to compute the second order we would want to keep various modes of Φ (which go like $S_n e^{-n\rho + in\theta}$) at the same time, so one might think that approximation of hyperbolic functions is not a good idea. However if we are interested in contribution to the second line of (C.6) which is proportional to $S_n S_m$, it is true that the leading contribution to this quantity is given by a Poincare patch result.

As we mentioned before, to construct the first perturbative correction to $AdS_5 \times S^5$ solution, one should start with harmonic function $\Phi^{(1)}$, then find the corresponding fields $\phi^{(1)}$, $g^{(1)}$, $h^{(1)}$ using equations (5.8), (5.5), then the fluxed can be recovered using (3.7)–(3.9). Now we solve this system in the Poincare patch, i.e. we consider the region of large ρ and replace hyperbolic functions by the exponents. Then it is convenient to start from a multipole expansion of the harmonic function $\Phi^{(1)}$:

$$\Phi^{(1)} = Q_0 \rho + \sum_{n>0} Q_n(z) e^{-n\rho} : (1 - z^2) \partial_z^2 Q_n - z \partial_z Q_n + n^2 Q_n = 0, \quad z \equiv \cos \theta. \quad (C.7)$$

Since we want this function to be suppressed compared to the $AdS_5 \times S^5$ contribution (which naively goes like e^ρ), the index n should be non-negative. However nontrivial Q_0 corresponds to changing the radius of the AdS space, so we will have to set it to zero. Also Q_1 corresponds to a dipole moment of the electrostatic problem, and since the total charge of the system is non-zero (it is related with radius of AdS), we can always make a shift in

x coordinate to set $Q_1(z) = 0$. Notice that there one can also add a constant to $\Phi^{(1)}$, but it will not affect the solution.

For large values of ρ , equation (5.8) becomes:

$$\frac{c}{s} \partial_y \left(\frac{s^3}{c} 2e^{-2\rho} g^{(1)} \right) = \frac{1}{4} \partial_y \left\{ \frac{s^2}{y} \partial_y \Phi^{(1)} - \frac{c^2}{x} \partial_x \Phi^{(1)} \right\} \approx \partial_y \left\{ e^{-2\rho} (y \partial_y - x \partial_x) \Phi^{(1)} \right\}. \quad (\text{C.8})$$

We observe that Q_0 does not source the correction to the metric $g^{(1)}$. Let us now introduce the mode expansion for the first order corrections:

$$g^{(1)} = \sum_{n>1} g_n(z) e^{-n\rho}, \quad h^{(1)} = \sum_{n>1} h_n(z) e^{-n\rho}, \quad \phi^{(1)} = \sum_{n>1} P_n(z) e^{-(n+2)\rho}. \quad (\text{C.9})$$

We introduced a shift into the modes of dilaton due to the relation (5.5) which implies that in the Poincare patch there is a linear relation between P_n and Q_n , g_n once expansions (C.9) are defined. For our purposes it would be convenient to express all fields in terms of P_n rather than Q_n but P_n can always be expressed in terms of Q_n using (C.8). In particular, (C.7) would imply a second order differential equation for P_n , but to get this equation one needs a certain amount of a guesswork. Rather than taking this route, we use a different method which directly leads to an equation for P_n , and we will use (C.8) to relate P_n and Q_n in the end.

In the approximation that we are taking, the equations for fluxes (3.7), (3.8) collapse to simple relations:

$$df_1^{(1)} = -e^{3\rho} [c d\phi^{(1)} - s * d\phi^{(1)}], \quad df_2^{(1)} = e^{3\rho} [c d\phi^{(1)} - s * d\phi^{(1)}] \quad (\text{C.10})$$

which imply that $f_1^{(1)} = f_2^{(1)}$. Substituting expansion of $\phi^{(1)}$ from (C.9), we find the expression for $f_1^{(1)}$:

$$df_1^{(1)} = - \sum_n e^{(1-n)\rho} \{ [-(n+2)cP_n + s^2 P_n'] d\rho + [-s c P_n' - (n+2)s P_n] d\theta \}. \quad (\text{C.11})$$

Integrability condition for this relation leads to a differential equation for P_n :

$$(1 - z^2)P_n'' - 5zP_n' - (4 - n^2)P_n = 0 \quad (\text{C.12})$$

and now we want to express everything in terms of this function. We begin with rewriting equation (5.8) in terms of $\phi^{(1)}$ rather than $g^{(1)}$:

$$-8\partial_y(s^2\phi^{(1)}) - \frac{8y\phi^{(1)}}{s^2 + sh^2} = \frac{4\partial_y\Phi^{(1)}}{s^2 + sh^2} + 2\partial_y \left[s^2 \frac{\partial_y\Phi^{(1)}}{y} + c^2 \frac{\partial_x\Phi^{(1)}}{x} \right]. \quad (\text{C.13})$$

Going to large values of ρ , we find an equation which holds on the Poincare patch:

$$-\partial_y(s^2\phi^{(1)}) - 2e^{-\rho}s\phi^{(1)} = 2e^{-2\rho}\partial_y\Phi^{(1)} + \partial_y \left[e^{-2\rho}(y\partial_y\Phi^{(1)} + x\partial_x\Phi^{(1)}) \right].$$

Expanding this equations in terms of modes, we can solve for Q_n as a function of P_n :

$$Q_n = \frac{1}{(n-1)n^2} [z(1-z^2)P_n' + ((2+n^2)(1-z^2) - 3)P_n]. \quad (\text{C.14})$$

To arrive at this relation we used equation (C.12). As a consistency check one can see that function Q_n satisfies its equation (C.7) as long as P_n satisfies (C.12)¹⁹ One can also invert a relation (C.14) to express P_n through Q_n and Q'_n , but the result is quite complicated, so we do not write it here. Finally we can use equations (C.4) to evaluate

$$g_n = h_n = \frac{n+1}{2n(n-1)} \left[-2z(1-z^2)P'_n - 2(n+2)(1-z^2)P_n + \frac{(n+2)(n+3)}{n+1}P_n \right]. \quad (C.15)$$

Once we have a complete result for the solution in the first order, it can be used to compute the second correction to Ψ_1 and Ψ_2 , in particular we want to check whether the simple algebraic relation similar to (C.1) persists at the second order. The computations are straightforward but tedious, so we present only the essential steps. We begin with looking at the following expression and expand up to second order:

$$\begin{aligned} \frac{e^{-\phi}}{e^{2A} - e^{2B}} \left[e^{A+\phi} F d\phi - y * d\phi \right] &= d(x\phi) - *d(y\phi) \\ &- \sum_{n>1} \frac{e^{(1-n)\rho} s}{4n} (2z(1-z^2)P'_n - (n+2)(2z^2+1)P_n) * d\phi \\ &+ \sum_{n>1} \frac{e^{(1-n)\rho}}{2} \left[\frac{(n+2)zP_n - (1-z^2)P'_n}{n-1} - \frac{z(n+2)(1+2z^2)P_n - 2z^2(1-z^2)P'_n}{2n} \right] d\phi. \end{aligned} \quad (C.16)$$

We now observe that there exists a set of harmonic functions ζ_n :

$$\zeta_n \equiv e^{(3-m)\rho} s \left[z(1-z^2)P'_n - ((n+1) + (2-n)z^2)P_n \right] \quad (C.17)$$

and we also functions $\tilde{\zeta}_n$ which are dual to ζ_n :

$$d\tilde{\zeta}_n = *d\zeta_n : \quad \partial_\rho \tilde{\zeta}_n = (1-n)\tilde{\zeta}_n = \partial_\theta \zeta_n. \quad (C.18)$$

Plugging this into the (C.16), we find

$$\begin{aligned} \frac{e^{-\phi}}{e^{2A} - e^{2B}} \left[e^{A+\phi} F d\phi - y * d\phi \right] &= d(x\phi) - *d(y\phi) \\ &+ \sum e^{(1-n)\rho} \left[\frac{\tilde{\zeta}_n}{2n} d\phi - \frac{\zeta_n}{2n} * d\phi + \left(\frac{z(3-4z^2)}{4} P_n d\phi - \frac{1-4z^2}{4} P_n * d\phi \right) \right] \\ &= d \left(x\phi + \sum \frac{e^{(1-n)\rho} \tilde{\zeta}_n}{2n} \phi \right) - *d \left(y\phi + \sum \frac{e^{(1-n)\rho} \zeta_n}{2n} \phi \right) \\ &+ \sum \frac{e^{(1-n)\rho}}{4} P_n \left[-(1-4z^2)s * d\phi + z(3-4z^2)d\phi \right]. \end{aligned} \quad (C.19)$$

Notice that due to the relations

$$\partial_\theta(z(3-4z^2)) = 3s(-1+4z^2), \quad \partial_\theta(s(1-4z^2)) = 3z(3-4z^2) \quad (C.20)$$

¹⁹The computation mentioned here can be easily performed in Mathematica.

we have the duality

$$d(z(3 - 4z^2)e^{3\rho}) = - * d[(-1 + 4z^2)e^{3\rho}] \tag{C.21}$$

this leads to equation

$$\begin{aligned} \frac{e^{-\phi}}{e^{2A} - e^{2B}} \left[e^{A+\phi} F d\phi - y * d\phi \right] &= d \left(x\phi + \sum \frac{e^{(1-n)\rho} \tilde{\zeta}_n}{2n} \phi + \frac{x\phi^2}{4} e^{2\rho} (3 - 4z^2) \right) \\ - * d \left(y\phi + \sum \frac{e^{(1-n)\rho} \zeta_n}{2n} \phi + \frac{y\phi^2}{4} e^{2\rho} (1 - 4z^2) \right). \end{aligned}$$

At this point we already have the expressions for $\Psi_1^{(2)}$ and $\Psi_2^{(2)}$, but it turns out that using explicit form of the ζ and $\tilde{\zeta}$ they can be rewritten in a simpler form

$$\begin{aligned} \Psi_1 &= y\phi \left(\frac{1}{e^{2A} - e^{2B}} + \sum \frac{e^{-n\rho}}{4} (4z^2 - 1) P_n \right) \\ &= y\phi \left(\frac{1}{e^{2A} - e^{2B}} + \frac{e^{2\rho}}{4} (4z^2 - 1) \phi^{(1)} \right) \\ \Psi_2 &= \phi \left(\frac{F e^A}{e^{2A} - e^{2B}} - \sum \frac{x P_n}{4} e^{-n\rho} (3 - 4z^2) \right) \\ &= \phi \left(\frac{F e^A}{e^{2A} - e^{2B}} - \frac{x e^{2\rho}}{4} \phi^{(1)} (3 - 4z^2) \right). \end{aligned} \tag{C.22}$$

The brackets in the above equations contain two terms: the first term suggests a simple algebraic relation analogous to (C.1), but the second terms destroy such simple connection. If one could guess the general structure of such extra terms (and if such terms can be written down in terms of algebraic functions or derivatives of warp factors) one would be able to start from a harmonic function Φ and write a solution of the entire system. Unfortunately, equation (C.22) seems to indicate that if such algebraic expressions for Ψ_1 and Ψ_2 exist, they would be quite complicated, so at present time we have to rely on perturbation theory to find the geometry.

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